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Analytical and approximate solutions to autonomous, nonlinear, third-order ordinary differential equations

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ABSTRACT

Analytical solutions to autonomous, nonlinear, third-order nonlinear ordinary differential equations invariant under time and space reversals are first provided and illustrated graphically as functions of the coefficients that multiply the term linearly proportional to the velocity and nonlinear terms. These solutions are obtained by means of transformations and include periodic as well as non-periodic behavior. Then, five approximation methods are employed to determine approximate solutions to a nonlinear jerk equation which has an analytical periodic solution. Three of these approximate methods introduce a linear term proportional to the velocity and a book-keeping parameter and employ a Linstedt–Poincaré technique; one of these techniques provides accurate frequencies of oscillation for all the values of the initial velocity, another one only for large initial velocities, and the last one only for initial velocities close to unity. The fourth and fifth techniques are based on the Galerkin procedure and the well-known two-level Picard's iterative procedure applied in a global manner, respectively, and provide iterative/sequential approximations to both the solution and the frequency of oscillation.

1. Introduction

Third-order, autonomous, nonlinear, ordinary differential equations are also known as jerk equations because the derivative of the acceleration with respect to time is referred to as the "jerk" [1,2], may have periodic or limit cycle solutions [3,4] and may exhibit chaos [5–7] because their phase space is three-dimensional.

Although, some authors [8,9] have in the past questioned whether there are any useful applications of third-order differential equations in physics based on the fact that most of the fundamental equations in physics are second-order ordinary or partial differential equations, e.g., Newton's second law, the Schrödinger equation, Einstein's field equations, etc., it must be noted that differentiation of a second-order ordinary differential equation with respect to the independent variable results in a (Newtonian) jerk equation which is of mathematical but not physical interest.

Mathematical studies of third-order nonlinear ordinary differential equations include those of Tunç [10] who proved the stability and boundedness of solutions of nonlinear vector differential equations by means of Lyapunov's second method, Ezeilo [11–13], Rao [14], Reissig et al. [15], Tunç and Ateş [16], etc. On the other hand, mathematical modelling of several physical phenomena sometimes result in third-order nonlinear ordinary differential equations. For example, Rauch [17] analyzed a nonlinear third-order ordinary differential equation that models the current in a vacuum tube circuit where the nonlinearities arise from the nonlinear characteristics of the tube, and investigated both periodic and quasi-periodic responses. Friedrichs [18] also considered models of vacuum tube circuits that are modelled by means of third-order ordinary differential equations, whereas Sherman [19] analyzed third-order ordinary differential equations that model the dynamics of nuclear spin generators.

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Third-order ordinary differential equations also appear in models of thermo-mechanical oscillators in fluids [22], transverse motions of piano strings [21], interactions between an elastic sphere and a surrounding fluid [20], vibrations of a mass attached to two horizontal strings and subject to aerodynamic forces [23], control systems [24,25], etc.

According to Gottlieb [3,4], the most general nonlinear jerk equation which is invariant under time- and displacement-reversals is

$$\dddot{x} = -\gamma\dot{x} - \alpha\dot{x}^3 - \beta x^2\dot{x} + \delta x\dot{x}\ddot{x} - \epsilon\dot{x}\ddot{x}^2, \tag{1}$$

where the dot denotes differentiation with respect to t and $\alpha, \beta, \gamma, \delta$ and ϵ are constants, and, at least, one of β, δ and ϵ should be different from zero. In addition, if $\epsilon = 0$, it is required that $\delta \neq -2\alpha$ so that the jerk equation is not the time-derivative of a second-order ordinary differential equation.

Gottlieb [3,4] provided solutions to Eq. (1) for $(\alpha, \beta, \gamma, \delta, \epsilon) = (0, 0, 1, 1, 0)$, $(0, 0, 1, 0, 1)$ and $(1, 1, 0, 0, 0)$ in his Examples 1–3, respectively, for $x(0) = 0$, $\dot{x}(0) = b$ and $\ddot{x}(0) = 0$. Gottlieb [3,4] also obtained an approximate periodic solution to Eq. (1) by means of a first-order harmonic balance procedure. Wu et al. [26] considered the third example treated by Gottlieb [3] and provided approximate solutions which were obtained by means of a linearized harmonic balance procedure, whereas Hu [27] also considered Gottlieb's Example 3 and obtained approximate solutions based on the use of parameter perturbation method which makes use of the Linstedt–Poincaré technique and expands both the solution and $\gamma = 0$ in Eq. (1) in terms of an artificial or book-keeping parameter. Ma et al. [28] have also considered Gottlieb's Example 3, introduced the linear stiffness term $\lambda\dot{x}$ with $\lambda = 1$ in both sides of Eq. (1) and obtained approximate solutions to both the solution and the frequency of oscillation by expanding $x(t)$ and λ in power series of an artificial parameter.

In this paper, some analytical solutions to Eq. (1) are first provided for the following initial conditions

$$x(0) = a, \qquad \dot{x}(0) = b, \qquad \ddot{x}(0) = c, \tag{2}$$

where a, b and c are constant, and for a variety of values of $(\alpha, \beta, \gamma, \delta, \epsilon)$. These analytical solutions are obtained by means of appropriate transformations and some result in the well-known Bernoulli and Riccati equations. The solutions reported here include those that diverge as time tends to infinity which may not be relevant in nonlinear dynamics, as well as periodic solutions, and complement the periodic solutions previously obtained by Gottlieb [3]. Some of these analytical solutions are illustrated in order to exhibit the effects of the parameters $(\alpha, \beta, \gamma, \delta, \epsilon)$ on the solution. The paper then presents five approximate solutions to Gottlieb's third example [3]. Three of these approximate techniques make use of an artificial parameter and the Linstedt–Poincaré method; the first of these techniques introduces both a term linearly proportional to the velocity in both sides of and an artificial parameter in Eq. (1) and works directly with the independent variable t, whereas the second one introduces a term linearly proportional to the velocity in both sides of the equation, a book-keeping parameter and a new independent variable. The third Linstedt–Poincaré method introduces a term linearly proportional to the velocity with a coefficient equal to one in both sides of the equation and an artificial parameter, and expands the solution in terms of this book-keeping parameter. The fourth and fifth techniques are based on Galerkin approximations and an iterative procedure, respectively.

The first Galerkin technique presented here is analogous to the first-order harmonic balance procedure employed by Gottlieb [3,4] while the second one makes use of the integral of Eq. (1). The fifth technique provides approximations to both the solution and the frequency of oscillation in an iterative (sequentially) manner as compared with the series solutions obtained with the Linstedt–Poincaré method. Even though, only first- and second-order approximations of the fourth and fifth techniques are presented in the paper, it must be noted that the objective of presenting them here is two-fold. First, it is shown that the first-order approximations of these techniques coincide with the first term of the series of the first two Linstedt–Poincaré methods. Second, the fourth and fifth techniques provide cumbersome nonlinear algebraic equations for the determination of the frequency of oscillation at higher-order and these equations must, in general, be solved numerically, whereas the three Linstedt–Poincaré procedures presented here provide explicit expressions for the terms of the series approximation to the frequency of oscillation.

The paper has been arranged as follows. In Section 2, some analytical solutions to Eq. (1) are obtained by means of transformations that result in Bernoulli and Riccati equations. In Section 2, some sample results of the analytical periodic solutions of Eq. (1) are also presented. Three Linstedt–Poincaré, two harmonic balance/Galerkin, and an iterative method are used to determine the approximate periodic solution of Eq. (1) for $\alpha = \beta = 1$ and $\gamma = \delta = \epsilon = 0$ and the results are compared with those of harmonic balance, linearized harmonic balance, and parameter perturbation techniques. A final section on the main findings reported in the paper puts an end to the paper.

2. Analytical solutions

Upon introducing $y \equiv \dot{x}$, Eq. (1) can be written as

$$(yy')' = -\gamma - \alpha y^2 - \beta x^2 + \delta xyy' - \epsilon y^2(y')^2, \tag{3}$$

where $y = 0$ has been disregarded, and $y' \equiv \frac{\mathrm{d}y}{\mathrm{d}x}$.