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Linear stability and Hopf bifurcation in a delayed two-coupled oscillator with excitatory-to-inhibitory connection☆

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We consider the dynamical behavior of a delayed two-coupled oscillator with excitatory-to-inhibitory connection. Some parameter regions are given for linear stability, absolute synchronization, and Hopf bifurcations by using the theory of functional differential equations. Conditions ensuring the stability and direction of the Hopf bifurcation are determined by applying the normal form theory and the center manifold theorem. We also investigate the spatio-temporal patterns of bifurcating periodic oscillations by using the symmetric bifurcation theory of delay differential equations combined with representation theory of Lie groups. Finally, numerical simulations are given to illustrate the results obtained.

1. Introduction

There are many papers devoted to the investigation of the dynamical behavior of two coupled neural oscillators (see, for example, [1,2]). The main functional unit of oscillatory neural networks is a neural oscillator. Some neural oscillator models consider oscillations as an endogenous property of a pacemaker neuron, such as the Van der Pol model [3], the Hindmarsh–Rose model [4], and the Hodgkin–Huxley model [5]. Another approach suggests that oscillations arise as a result of the interactions between neural populations, for example, between excitatory and inhibitory populations, such as the Wilson–Cowan model [6], *integrate and fire* model [7], and McGregor model [8].

The model of a single neural oscillator is represented by a system of two autonomous differential equations describing the dynamics of average activities of the excitatory and the inhibitory populations (measured as a portion of firing neurons in each population). Denoting these activities by E and I, respectively, the model reads:

$$\begin{cases} E'(t) = -E(t) + af(-c_1 I(t-\tau)), \\ I'(t) = -I(t) + af(c_2 E(t-\tau)), \end{cases} \tag{1}$$

where a, c_1, c_2 and τ are positive constants, and f is the activation function, which is usually adopted as $f(x) = \tanh x$. Throughout this paper, we always assume that $f: \mathbb{R} \to \mathbb{R}$ is $\mathcal{C}^1$-smooth function with $f(0) = 0$. Without loss of generality, we also assume that $f'(0) = 1$. For a number of two neuron models and their linear stability analysis, we refer to the works of Marcus et al. [9], Babcock and Westervelt [10,11], Gopalsamy and Leung [12]. Consider two identical neural oscillators

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given by (1) and which are coupled using terms that may be interpreted as an additional external input. This system of coupled oscillators has the form

$$\begin{cases} E_1'(t) = -E_1(t) + af(-c_1 I_1(t-\tau) + P_{12}(t-\tau)), \\ I_1'(t) = -I_1(t) + af(c_2 E_1(t-\tau) + Q_{12}(t-\tau)), \\ E_2'(t) = -E_2(t) + af(-c_1 I_2(t-\tau) + P_{21}(t-\tau)), \\ I_2'(t) = -I_2(t) + af(c_2 E_2(t-\tau) + Q_{21}(t-\tau)), \end{cases} \tag{2}$$

where $P_{12} = \alpha_1 E_2 - \alpha_2 I_2$, $P_{21} = \alpha_2 E_1 - \alpha_1 I_1$, $Q_{12} = \alpha_3 E_2 - \alpha_4 I_2$, $Q_{21} = \alpha_3 E_1 - \alpha_4 I_1$. Here (E_1, I_1) and (E_2, I_2) describe the activities of the first and the second oscillators, respectively. The terms $P_{12}, P_{21}, Q_{12}, Q_{21}$ describe connections between the oscillators, and coefficients $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ represent the strength of the connections between sub-populations related to different oscillators. Since we have symmetric coupling of identical oscillators, Eq. (2) has the reflection symmetry of interchange of two oscillators.

In this paper, we consider the connections from the excitatory population of one oscillator to the inhibitory population of the other oscillator, i.e.,

$$P_{12} = P_{21} = 0, \qquad Q_{12} = \alpha E_2, \qquad Q_{21} = \alpha E_1,$$

where α is the control parameter. Namely,

$$\begin{cases} E_1'(t) = -E_1(t) + af(-c_1 I_1(t-\tau)), \\ I_1'(t) = -I_1(t) + af(c_2 E_1(t-\tau) + \alpha E_2(t-\tau)), \\ E_2'(t) = -E_2(t) + af(-c_1 I_2(t-\tau)), \\ I_2'(t) = -I_2(t) + af(c_2 E_2(t-\tau) + \alpha E_1(t-\tau)). \end{cases} \tag{3}$$

Here we are interested in studying how the coupling strength and the time delay can affect the stability of the zero solution of system (3) and the bifurcations of periodic solutions when stability is lost.

Note that the coupling is identical and this leads to a nonlinear system with reflection $\mathbb{Z}_2$ symmetry. Thus, it is natural to consider the effect of the coupling strength and the time delay on the existence, spatio-temporal pattern, and stability of Hopf bifurcating periodic solutions in (3). Golubitsky and Stewart [13] provide abstract techniques, based on group theory, for predicting the occurrence of particular combinations of spatial and temporal symmetries when a symmetric system of ordinary differential equations undergoes Hopf bifurcation. For delay differential equations, the general symmetric Hopf bifurcation theorem has been developed by Wu [14], Guo and Lamb [15]. In short, their theorems assert that at a symmetric analogue of a Hopf bifurcation, one or more branches of periodic solutions bifurcate. These oscillations may be distinguished by their spatiotemporal symmetry groups Σ, which are subgroups of $\mathbb{Z}_2 \times \mathbb{S}^1$. Here, $\mathbb{S}^1$ is the circle group and it acts on a periodic solution by phase shift. The isotropy subgroups measure the amount of symmetry present in the branching solutions. The question of the existence of symmetry-breaking oscillations is thus reduced to purely group-theoretic calculations and depends only on the symmetry assumed in the system. The main point here is that typical oscillation patterns of a system can be predicted in terms of its symmetries, without investigating the detailed dynamical equations. As such, our model (3) contributes an additional example of bifurcation computation in delay differential equations with $\mathbb{Z}_2$ symmetry in the context of artificial neural networks.

The rest of this paper is organized as follows: In Section 2, we discuss the associated characteristic equation, the linear stability of the trivial solution and the synchronous pattern of solution. Section 3 is devoted to the spatio-temporal pattern, bifurcation direction and stability of the Hopf bifurcating periodic solutions. Patterns of nontrivial equilibria and numerical simulations are given in Section 4. Section 5 lists all of the possible codimension two bifurcations.

2. Linear stability analysis

Let $\mathscr{C} = C([-\tau, 0], \mathbb{R}^4)$ denote the Banach space of all continuous mappings from $[-\tau, 0]$ into $\mathbb{R}^4$ equipped with the supremum norm $\|\phi\| = \sup_{\theta\in[-\tau,0]} |\phi(\theta)|$ for $\phi \in \mathscr{C}$. As usual, if $\sigma \in R$, $A \geq 0$ and $u : [\sigma-\tau, \sigma+A] \to \mathbb{R}^4$ is a continuous mapping, then $u_t \in \mathscr{C}$ for $t \in [\sigma, \sigma+A]$ is defined by $u_t(\theta) = u(t+\theta)$ for $-\tau \leq \theta \leq 0$.

Linearizing system (3) at the trivial solution leads to the following linear system:

$$\begin{cases} E_1'(t) = -E_1(t) - ac_1 I_1(t-\tau), \\ I_1'(t) = -I_1(t) + ac_2 E_1(t-\tau) + a\alpha E_2(t-\tau), \\ E_2'(t) = -E_2(t) - ac_1 I_2(t-\tau), \\ I_2'(t) = -I_2(t) + ac_2 E_2(t-\tau) + a\alpha E_1(t-\tau). \end{cases} \tag{4}$$

Then, the characteristic matrix for system (4) is

$$\Delta(\lambda) = (\lambda+1)\mathrm{Id} - M\mathrm{e}^{-\lambda\tau}, \tag{5}$$

where

$$M = \begin{pmatrix} 0 & -ac_1 & 0 & 0 \\ ac_2 & 0 & a\alpha & 0 \\ 0 & 0 & 0 & -ac_1 \\ a\alpha & 0 & ac_2 & 0 \end{pmatrix}$$