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Singular spectrum analysis based on the minimum variance estimator

Hossein Hassani

Institute for Studies and Research (ITSR), Tehran, Iran

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ABSTRACT

In recent years Singular Spectrum Analysis (SSA), used as a powerful technique in time series analysis, has been developed and applied to many practical problems. In this paper, the SSA technique based on the minimum variance estimator is introduced. The SSA technique based on the minimum variance and least squares estimators in reconstructing and forecasting time series is also considered. A well-known time series data set, namely, monthly accidental deaths in the USA time series, is used in examining the performance of the technique. The results are compared with several classical methods namely, Box–Jenkins SARIMA models, the ARAR algorithm and the Holt–Winter algorithm.

1. Introduction

It is well known that errors can seriously limit the performance of the methods and techniques. Effective methods for dealing with noisy data, especially noisy time series are currently still lacking. There are several noise reduction methods. It is accepted that singular value decomposition (SVD) based methods and signal subspace (SS) methods are more effective than many others for noise reduction and forecasting in financial and economic time series [1].

Having a method for decomposing the vector space of the noisy time series into a subspace that is generated by the noise free series and a subspace for the noise series, we can construct the noise free time series. The approximate decomposition of the vector space of the noisy time series into noise free time series and noise series subspaces can be done with, for example, the orthogonal matrix factorization technique such as SVD.

The idea to perform the SS method was proposed in [2] where a modified SVD is used for the reconstruction of noise free series. A general framework for recovering noise free series has been presented in [3]. The method which forms the basis for a very general class of subspace based noise reduction algorithms, is based on the assumption that the original time series exhibits some well-defined properties or obeys a certain model. Noise free series is therefore obtained by mapping the original time series onto the space of series that possess the same structure as the noise free series.

In this context, the Singular Spectrum Analysis (SSA) technique, which is a SVD and SS based method, can be considered as a proper method for noise reduction and forecasting time series data sets. The SSA technique incorporates the elements of classical time series analysis, multivariate statistics, multivariate geometry, dynamical systems and signal processing. The aim of SSA is to make a decomposition of the original series into the sum of a small number of independent and interpretable components such as a slowly varying trend, oscillatory components and a structureless noise.

The appearance of SSA is usually associated with the publication of [4]. Possible application areas of SSA are diverse [5]: from mathematics and physics to economics and financial mathematics, from metrology and oceanology to social science and market research (see, for example, [5–14] and the references therein). Any seemingly complex series with a potential structure could provide an example of a successful application of SSA. A thorough description of the theoretical and practical foundations of the SSA technique (with several examples) can be found in [5,15]. An elementary introduction to the subject can be found in [16].

E-mail address: hassanih@cf.ac.uk.

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All the aforementioned research is based on the standard SVD and the least squares (LS) estimate. The LS estimate of the noise free series can be obtained by truncating the singular values. The LS estimator projects the noisy time series onto the perturbed signal (noise+signal) subspace. The reconstructed series using LS estimator has the lowest possible (zero) signal distortion and the highest possible residual noise level. In this paper, we consider an alternative method which is based on the minimum variance (MV) estimator for reconstruction and forecasting noisy time series. The MV estimator is the optimal linear estimator, which gives the minimum total residual power [17,18].

The structure of the paper is as follows. Section 2 briefly describes least squares and minimum variance estimators. The reconstruction and forecasting algorithm is presented in Section 3. Our forecast results are then presented and described in Section 4 and some conclusions are given in Section 5.

2. LS and MV estimators

Consider a noisy signal vector $Y_T = (y_1, \ldots, y_T)'$ of length T. We will add the additive white noise to the noise free series (signal) and assume that the noise is uncorrelated with the signal:

$$Y_T = S_T + N_T; \tag{1}$$

here S_T represents the signal component and N_T the noise component. Let $K = T - L + 1$, where L is some integer called the window length (we can assume that $L \leq T/2$). Define the so-called 'trajectory matrix' $\mathbf{X} = (x_{ij})_{i,j=1}^{L,K}$, where $x_{ij} = y_{i+j-1}$. Note that $\mathbf{X}$ is a Hankel matrix (by the definition, these are the matrices such that their (i, j)th entries depend only on the sum $i + j$);

$$\mathbf{X} = (x_{ij})_{i,j=1}^{L,K} = \begin{pmatrix} y_1 & y_2 & y_3 & \cdots & y_K \\ y_2 & y_3 & y_4 & \cdots & y_{K+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_L & y_{L+1} & y_{L+2} & \cdots & y_T \end{pmatrix}. \tag{2}$$

We then consider $\mathbf{X}$ as a multivariate data with L characteristics and $K = T - L + 1$ observations. The columns X_j of $\mathbf{X}$, considered as vectors, lie in an L-dimensional space $\mathbb{R}^L$. It is obvious that:

$$\mathbf{X} = \mathbf{S} + \mathbf{N}, \tag{3}$$

where $\mathbf{S}$ and $\mathbf{N}$ represent Hankel matrices of the signal S_T and noise N_T, respectively. The singular value decomposition (SVD) of the trajectory matrix $\mathbf{X}$ can be written as:

$$\mathbf{X} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}', \tag{4}$$

where $\mathbf{U} \in \mathbb{R}^{L\times K}$ is the matrix that consists of the normalized eigenvector U_i corresponding to the eigenvalue λ_i ($i = 1, \ldots, L$), $\mathbf{V} \in \mathbb{R}^{K\times K}$, is the matrix that contains the principal components defined as $V_i = \mathbf{X}'U_i/\sqrt{\lambda_i}$, and $\boldsymbol{\Sigma} = \mathrm{diag}(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_L)$. The diagonal elements of $\boldsymbol{\Sigma}$ are called singular values of $\mathbf{X}$, and their set is called the singular value spectrum.

The SS methods are based on the assumption that the vector space of the noisy time series (signal) can be split into mutually orthogonal noise and signal+noise subspaces. The components in the noise subspace are suppressed or even removed completely. Therefore, one can reconstruct the noise free series from the signal+noise subspace by choosing the weight. Thus, by adapting the weights of the different singular components, an estimate of the Hankel matrix $\mathbf{X}$, which corresponds to noise reduced series, can be achieved:

$$\mathbf{X} = \mathbf{U}(\boldsymbol{W}\boldsymbol{\Sigma})\mathbf{V}', \tag{5}$$

where $\mathbf{W}$ is the diagonal matrix containing the weights. Now, the problem is choosing the weight matrix $\mathbf{W}$. Next we consider the problem of choosing this matrix using different criteria. The SVD of the matrix $\mathbf{X}$ can be written as:

$$\mathbf{X} = [\mathbf{U}_1 \quad \mathbf{U}_2] \begin{bmatrix} \boldsymbol{\Sigma}_1 & 0 \\ 0 & \boldsymbol{\Sigma}_2 \end{bmatrix} \begin{bmatrix} \mathbf{V}_1' \\ \mathbf{V}_2' \end{bmatrix} \tag{6}$$

where $\mathbf{U}_1 \in \mathbb{R}^{L\times r}$, $\boldsymbol{\Sigma}_1 \in \mathbb{R}^{r\times r}$ and $\mathbf{V}_1 \in \mathbb{R}^{K\times r}$. We can also represent the SVD of the Hankel matrix of the signal $\mathbf{s}_T$ as:

$$\mathbf{S} = \left[\mathbf{U}_{1s} \quad \mathbf{U}_{2s}\right] \begin{bmatrix} \boldsymbol{\Sigma}_{1s} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{V}_{1s}' \\ \mathbf{V}_{2s}' \end{bmatrix}. \tag{7}$$

It is clear that the Hankel matrix $\mathbf{S}$ cannot be reconstructed exactly if it is perturbed by noise.