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A generalization of the Haddock conjecture and its proof

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Abstract

In this paper, we study the asymptotic behavior of solutions for a class of neutral delay differential equations. These equations have important practical applications and generalize those on which Haddock conjectured that every solution of the equations tends to a constant. Our results improve the existing ones in the literature.

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1. Introduction

In the 1987 international conference on nonlinear system and their applications, Haddock [5] proposed the following conjecture:

Every solution of the neutral delay differential equation

$$(x(t) - cx(t-r))' = -ax^{\gamma}(t) + ax^{\gamma}(t-r) \tag{1.1}$$

tends to a constant as $t \to +\infty$, where $r > 0$, $0 < c < 1$, $a > 0$ and γ is a quotient of positive odd integers.

To confirm the above conjecture, Wu [12] considered the following equation:

$$(x(t) - cx(t-r))' = -F(x(t)) + F(x(t-r)), \tag{1.2}$$

where $r > 0$, $0 \leqslant c < 1$, and $F : R^1 \to R^1$ is a strictly increasing continuous function. Eq. (1.2) has been extensively studied (see, e.g. [1–4,6–14]) because of its applications in modeling population growth, the spread of epidemics, the compartmental system with pipes and so on. It should be mentioned that Eq. (1.2) is also considered in [9,10] when different delays appear on the two sides of (1.2). In particular, Krisztin [10] gives an example for a bounded and nonconvergent solution.

The purpose of this paper is to study the asymptotics of the following neutral delay differential equation:

$$(x(t) - cx(t-r))' = -F(x(t)) + G(x(t-r)), \tag{1.3}$$

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where $r>0$, $0<c<1$ and $F, G : R^1 \to R^1$ are continuous satisfying either $G(x) \geqslant F(x)$ for all $x \in R^1$ or $G(x) \leqslant F(x)$ for all $x \in R^1$. Note that no similar results exist for Eq. (1.3) with F being assumed to be nondecreasing only. In this paper, we show that if F is nondecreasing, then every bounded solution of (1.3) tends to a constant as $t \to +\infty$. Our approach is quite different from those of [1–4,6–14]. Moreover, our conditions are weak.

2. Preliminary results

In this section, we give some basic properties enjoyed by (1.1). For this purpose, we first introduce some notations.

In the sequel, $R^1(R^1_+)$ denotes the set of all (nonnegative) real numbers. Let $r>0$ be a given real number, and $C = C([-r, 0], \mathbf{R}^1)$ equipped with the supremum norm. Define $C_+ = C([-r, 0], R^1_+)$ and $K = \{\varphi \in C_+ : \varphi(0) - c\varphi(-r) \geqslant 0\}$. It follows that (C, C_+) and (C, K) are ordered Banach spaces with $c \in [0, 1)$. For any $\varphi, \psi \in C$ and a subset $A \subseteq C$, we write $\varphi \leqslant \psi$ iff $\psi - \varphi \in C_+$, $\varphi < \psi$ iff $\varphi \leqslant \psi$ and $\varphi \neq \psi$, $\varphi \ll \psi$ iff $\psi - \varphi \in \operatorname{Int} C_+$, $\varphi \leqslant_K \psi$ iff $\psi - \varphi \in K$, $\varphi <_K \psi$ iff $\varphi \leqslant_K \psi$ and $\varphi \neq \psi$, $\varphi \ll_K \psi$ iff $\psi - \varphi \in \operatorname{Int} K$, $\varphi \leqslant_K A$ iff $\varphi \leqslant_K \psi$ for all $\psi \in A$, $\varphi <_K A$ iff $\varphi <_K \psi$ for all $\psi \in A$, $\varphi \ll_K A$ iff $\varphi \ll_K \psi$ for all $\psi \in A$. Similarly, we can define $\psi \geqslant \varphi$, $\psi \geqslant_K \varphi$, $\varphi \geqslant_K A$, etc. If $\sigma \geqslant 0$ and $x \in C([-r, \sigma], R^1)$, then for any $t \in [0, \sigma]$, $x_t \in C$ is defined by $x_t(\theta) = x(t+\theta)$, $-r \leqslant \theta \leqslant 0$. For $\alpha \in R^1$, we define $\widehat{\alpha} \in C$ by $\widehat{\alpha}(\theta) = \alpha$ for all $\theta \in [-r, 0]$. Moreover, for $\varphi \in C$, we use $x_t(\varphi)$ $(x(t, \varphi))$ to denote the solution of (1.3) with the initial data $x_0(\varphi) = \varphi$.

In the following, it will be assumed, unless otherwise stated, that c belongs to $(0, 1)$.

Remark 2.1. Assume that $\varphi \in C$ and $\alpha \in R^1$ such that $\varphi \geqslant_K \hat{\alpha}$. Then the following statements are true:

(i) If $\varphi(0) - c\varphi(-r) = (1-c)\alpha$ and $\max\{\varphi(0), \varphi(-r)\} > \alpha$, then $\varphi(0) < \varphi(-r)$.
(ii) If $\varphi(0) - c\varphi(-r) = (1-c)\alpha$ and $\min\{\varphi(0), \varphi(-r)\} = \alpha$, then $\varphi(0) = \varphi(-r) = \alpha$.
(iii) If $\varphi(0) - c\varphi(-r) > (1-c)\alpha$, then $\varphi(0) > \alpha$.

Remark 2.2. Assume that $x \in C([-r, r], R^1)$ and $\alpha \in R^1$ such that $x_0 \geqslant_K \hat{\alpha}$. Then the following statements are true:

(i) If $x(t) - cx(t-r) \geqslant (1-c)\alpha$ for all $t \in [0, \tau]$, then $x_t \geqslant_K \hat{\alpha}$ for all $t \in [0, \tau]$.
(ii) If $x(t) - cx(t-r) > (1-c)\alpha$ for all $t \in [0, r]$, then $x_r \gg_K \hat{\alpha}$.

Furthermore, for the sake of convenience, we always assume that $F, G : \mathbf{R}^1 \to \mathbf{R}^1$ are continuous and $F(x)$ is nondecreasing on $\mathbf{R}^1$ in the rest of this paper.

Lemma 2.1 (*Yi and Huang [14, Lemma 3.1]*). *Let $F \in C(R^1)$ be nondecreasing on R^1. For any constants K, t_0 and x_0, the initial value problem*

$$\begin{cases} x'(t) = -F(x(t)) + K, \\ x(t_0) = x_0 \end{cases} \tag{2.1}$$

has a unique solution $x(t, t_0, x_0)$ on $[t_0, \infty)$.

By an argument similar to that of Lemma 3.2 in [14], we can obtain the following Lemma 2.2.

Lemma 2.2. *Let r be a positive constant, $a, b \in C([t_0, t_0+r])$ and $F \in C(R^1)$ be nondecreasing on R^1. Then for any constant $x_0 \in R^1$, the initial value problem*

$$\begin{cases} x'(t) = -F(x(t) + a(t)) + b(t), \\ x(t_0) = x_0 \end{cases} \tag{2.2}$$

has a unique solution $x(t; t_0, x_0)$ on $[t_0, t_0+r]$.

By a straightforward induction argument, we can then apply Lemma 2.2 to obtain the following result.