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Unsteady flow of a second grade fluid between two side walls perpendicular to a plate

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Abstract

Exact solutions for the unsteady flow of a second grade fluid induced by the time-dependent motion of a plane wall between two side walls perpendicular to the plane are established by means of the Fourier sine transforms. The similar solutions for Newtonian fluids, performing the same motions, are obtained as limiting cases for $\alpha_1 \to 0$. The steady solutions, the same for Newtonian and non-Newtonian fluids, are also obtained as limiting cases for $t \to \infty$. In the absence of the side walls, all solutions that have been obtained reduce to those corresponding to the motion over an infinite plate. Graphical illustrations show that the diagrams corresponding to the velocity field in the middle of channel and the shear stress at the bottom wall for a second grade fluid are going to be those for a Newtonian fluid if the normal stress module $\alpha_1 \to 0$.

Keywords: Unsteady flow; Second grade fluid; Exact solutions; Flat plate; Side walls

1. Introduction

The theory of non-Newtonian fluids has received much attention recently, because the traditional Newtonian fluids cannot precisely describe the characteristics of most of the fluids in industry and technology. Mathematically such fluids have a non-linear relationship between the stress and the rate of strain at a point. These fluids have been modeled in a number of diverse manners with their constitutive equations varying greatly in complexity. Amongst the various types of non-Newtonian fluids, the viscoelastic fluids are most commonly used. This is because of their frequent occurrence in disparate processes in engineering, science and biology, for example, in polymer processing, coating, ink-jet printing, microfluidics, geological flows in the earth mantle, hemodynamics, flow of synovial fluid in joints, and many others. In general for differential type fluids of grade n, the governing equations of motion are of order $(n+1)$. The adherence boundary conditions are insufficient for determinacy when $n > 1$. Therefore, for a unique solution, one needs the additional boundary conditions. The detailed critical review on the boundary conditions, the existence and uniqueness of the solution has been given by Rajagopal [23,26], Rajagopal et al. [29] and Rajagopal and Kaloni [28]. The governing differential equations for non-Newtonian fluids are also highly non-linear in general and thus the solutions are more

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difficult to obtain. This is not only true of exact, approximate analytic solutions but even of numerical solutions. There is a simplest subclass of viscoelastic differential type fluids namely the second grade for which one can reasonably hope to obtain the analytic solutions. Motivated by this fact, several researchers [3,5,9–13,16–22,24,32] are engaged in studying the flows of second grade fluids.

In this paper, our concern is to investigate the unsteady flow of a second grade fluid induced by the time-dependent motion of a plane wall between two side walls perpendicular to the plane. It is also known that for the linear differential equations, the Laplace transform method is suited in obtaining solutions for small times. But, it is not a trivial matter to invert the Laplace transform. Bandelli et al. [4] already analyzed that for Rayleigh problem, the obtained solution by Laplace transform does not satisfy the initial condition. They showed that this is due to an incompatibility between the prescribed data. For a comprehensive discussion on this issue, we refer the readers to the work of Bandelli [2]. Due to this fact in mind the exact solutions here are established by means of Fourier sine transforms. For $\alpha_1 \to 0$ they reduce to the similar solutions for Newtonian fluids while for $h \to \infty$ the solutions corresponding to the motion over an infinite plate are recovered. The variation of the velocity field in the middle of the channel and the shear stress at the bottom wall are finally illustrated for several values of the material constants.

2. Governing equations

The Cauchy stress $\mathbf{T}$ in an incompressible homogenous fluid of second grade is related to the fluid motion in the following manner [33]

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{1}$$

where $p\mathbf{I}$ is the indeterminate part of the stress due to the constraint of incompressibility, $\mathbf{A}_1$ and $\mathbf{A}_2$ are the first two Rivlin–Ericksen tensors, μ is the dynamic viscosity and α_1 and α_2 are normal stress moduli. If the model (1) is required to be compatible with thermodynamics in the sense that all motions satisfy the Clausius–Duhem inequality and the assumption that the specific Helmholtz free energy is a minimum in equilibrium, then the material moduli must meet the following restrictions [6,25]:

$$\mu \geqslant 0, \quad \alpha_1 \geqslant 0 \quad \text{and} \quad \alpha_1 + \alpha_2 = 0. \tag{2}$$

The sign of the material moduli α_1 and α_2 has been the subject of much controversy [7]. A comprehensive discussion on the restrictions given by (2) as well as a critical review on the fluids of differential type can be found in an exhaustive work by Dunn and Rajagopal [7].

The equation of motion of incompressible second grade fluids is one order higher than the Navier–Stokes equation. As a result the no-slip boundary condition for a second grade fluid may not be sufficient here and therefore, in order to solve a well-posed problem, one needs an additional boundary/initial condition [27].

In this paper, unsteady unidirectional flows of a fluid of second grade are considered. The velocity field is assumed to be in the following form:

$$u = u(y, z, t), \quad v = 0, \quad w = 0 \tag{3}$$

in the system of Cartesian coordinates x, y and z. For these flows the constraint of incompressibility is automatically satisfied and the governing equation is [15]

$$(v + \alpha\partial_t)(\partial_y^2 + \partial_z^2)u(y, z, t) = \partial_t u(y, z, t), \tag{4}$$

where $v = \mu/\rho$ is the kinematic viscosity of the fluid and $\alpha = \alpha_1/\rho$, ρ being the constant density of the fluid.

3. Flow induced by the time-dependent motion of a flat plate between two side walls perpendicular to the plate

Suppose that a second grade fluid occupies the space above a plane wall perpendicular to the y-axis and between two side walls situated in the planes $z = 0$ and d. Initially, the fluid is at rest and at time $t = 0^+$ it is suddenly set into motion by translating the bottom wall in its plane with the time-dependent velocity $U(t)$. Its velocity is of the form (3)