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An alternative mathematical algorithm for the photo- and videokeratoscope

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Abstract

Due to the resolution of current laser technology, the accuracy of corneal topography as measured by the videokeratoscope is no longer adequate to provide precise enough data for refractive surgery or for the fitting of customized contact lenses. We present an algorithm for recovering corneal topography that makes use of modern differential geometric techniques and numerical descent in Sobolev spaces. We believe this algorithm may be used with the photo- and videokeratoscope to increase the accuracy of the recovered corneal topography.

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1. Introduction

Accurate measurements of optical power (curvature) are required for successful refractive surgery techniques and for the fitting of customized contact lenses in order to address both higher- and lower-order optical abberations such as astigmatism, coma, hyperopia, myopia, and spherical abberations. The study of corneal topography dates to the 1880s when Plácido [26] introduced what became known as the Plácido disk. The development of the photokeratoscope and videokeratoscope advanced these techniques by merging modern computer technology with time-proven techniques. See [19,11] for thorough reviews of the devices and the methods utilized by such devices for measuring corneal topography. See [2] for a discussion of the method of the alternative wave front (HS) technology for determining corneal aberrations. Turuwhenua [33] states that the videokeratoscope is still widely used in clinical practice and provides references for the commercially available devices [7,8,36,32], the principles of operation of such devices [5,35,34], and the clinical applications [29,9,31,18].

According to Barbosa [19], accuracy in dioptric power of .1D to .25D is to be expected. Given that accurate information concerning irregularities of the cornea is a first step toward improved contact lens manufacturing and laser surgery techniques, we offer an alternative algorithm that recovers a simulated curve to an accuracy of 10^{-5} and optical power to an accuracy of .0493 diopters.

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The algorithm uses differential geometry based on the work of Oliker [23,24,22] to produce a differential equation in terms of the optical path length. Solving the differential equation using Sobolev steepest descent, we recover the optical path length, the curve, and the dioptric power to a high degree of accuracy. Sobolev steepest descent is a systematic preconditioning technique where gradients are based on Sobolev spaces rather than on Euclidean space, yielding superior results in terms of both time and accuracy. Introduced by Neuberger in [20], a complete discussion may be found in [21]. Problem-specific applications are given in [30,4,6,10,17,28] and general references for Sobolev spaces in [1,13]. In [15] a convergence proof is given for discrete spaces such as those in this paper. For a paper concerning Sobolev gradients which are constructed based on the problem at hand, consider [16,27]. For a historical perspective on descent techniques, we direct the reader to [3] and for a general discussion consider [14,25].

2. The problem

We demonstrate our algorithm on the cylindrical target model as developed by Knoll [12], although the mathematics we present applies equally to the planar, conical, and hemispherical ring-target models described in [19]. A patient sits at the device while a cylinder of slightly larger radius than that of the human eye is placed over the eye. Illustrated in Fig. 1, this cylinder has multiple rings or slits at varying heights along its periphery. Light is projected through these slits onto the cornea, then reflected through a lens at the base of the cylinder and onto a planar surface where it forms images which are circular in shape. These ring images are used to recover the shape and curvature of the cornea. We reduce the problem to two dimensions and consider the problem of recovering curves in the plane which corresponds to taking a slice of the cylinder. Place the cylinder in 3-space such that the origin is at the center of the lens and the z-axis passes through the center of the cylinder. Consider the plane, illustrated in Fig. 2, with the x-axis contained in the base of the cylinder, the origin at the center of the lens, and the y-axis passing through the apex of the surface. The intersection of the cornea and the plane is now a two-dimensional curve. If we may recover the curve for this plane then we may replace the plane with one which is rotated slightly, and recover the resulting curve. In this manner, we may recover the entire surface.

It is noteworthy that the mathematics applies in higher dimensions, allowing us to generate a partial differential equation and recover the full surface in one step. Returning to our planar model, for each ring we assume that we know its height and the angle which the light ray emanating from that ring makes with the x-axis as it passes through the lens. This information is easily derived from the recorded ring images. From this information, we wish to recover the curve.

We make the following assumptions in 3-space.

(1) The surface has a normal at every point.
(2) The normal to the surface, the incident ray, and the reflected ray lie in the same plane.
(3) The angle between the normal and the incident ray equals the angle between the normal and the reflected ray.
(4) If p represents a point source of light on a ring of the cylinder and q represents its image on the planar surface then the function which assigns p to q is one to one.
(5) The reflected rays pass through the center of the lens.

In the plane, these assumptions may be restated.

(1) The curve has a tangent at every point.
(2) The angle between the incident ray and the tangent equals the angle between the reflected ray and the tangent.
(3) If p is a point on the line $x = d$ and α is the angle that the reflected ray makes with the x-axis as it passes through the origin, then the function which assigns α to p is one to one.
(4) The reflected ray passes through the origin.

For each ring we know its height, s, and the angle, α, the reflected ray makes with the x-axis as it passes through the lens (the origin). The method hinges on solving a differential equation to obtain the optical path length of the incident and reflected rays. The curve will be recovered from the optical path length.