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Validity of the resonant four-wave interaction system in a model for surface water waves on an infinite deep sea

Wolf-Patrick Düll*, Guido Schneider

Mathematisches Institut I, Universität Karlsruhe, 76128, Karlsruhe, Germany

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Abstract

Models for the resonant four-wave interaction play a big role in the description of gravity-driven surface water waves. The question however, if such models, really describe the evolution of surface water waves is a difficult task, according to the fact that the original water wave problem contains quadratic terms and a resonance at the wave vector $k=0$. As a first step in answering the validity question positively we prove here that solutions of a model for surface water waves on an infinite deep sea really can be described by the associated systems for the resonant four wave interaction. We believe that the method developed in this paper applies to the problem of surface water waves on an infinite deep sea, too.

Keywords: Water wave problem; FWI systems

1. Introduction and the result

Systems for the resonant four-wave interaction (FWI systems) describe the interaction of N modulated wave packets

$$\varepsilon A_j(\varepsilon^2 x, \varepsilon^2 t) e^{i(k_j x + \omega_j t)} \quad (j \in I_N = \{-N, \ldots, -1, 1, \ldots, N\}),$$

with $0 < \varepsilon \ll 1$ a small perturbation parameter, $A_j(X,T) \in \mathbb{C}$ with $A_j = \overline{A_{-j}}$, $k_j \in \mathbb{R}^2$ the spatial wave vector, and $\omega_j \in \mathbb{R}$ the temporal wave number, which are related through the linear dispersion relation. We consider the case that there are no quadratic resonances; i.e. there are no wave vectors $k_j \in \mathbb{R}^2$ and temporal wave numbers $\omega_j \in \mathbb{R}$ with

$$k_1 + k_2 + k_3 = 0 \quad \text{and} \quad \omega_1 + \omega_2 + \omega_3 = 0,$$

except one k_j equals zero.

In their simplest form the equations for the A_j look like

$$\partial_T A_j = c_j \partial_X A_j + \sum_{l \in I_N} d_{jl} |A_l|^2 A_j + \sum_{(j_1, j_2, j_3) \in R_j} d^j_{j_1 j_2 j_3} \overline{A_{j_1} A_{j_2} A_{j_3}},$$

* Corresponding author. Tel.: +497216082056; fax: +497216086214.
E-mail addresses: duell@math.uni-karlsruhe.de (W. Düll), guido.schneider@mathematik.uni-karlsruhe.de (G. Schneider).

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with the negative group velocities $c_j \in \mathbb{R}$, and purely imaginary coefficients d_{jl} and $d^j_{j_1 j_2 j_3}$. The set R_j is called the set of resonances and it is defined by

$$R_j = \{(j_1, j_2, j_3) \in I_N^3 \mid k_{j_1} + k_{j_2} + k_{j_3} + k_j = 0, \quad \omega_{j_1} + \omega_{j_2} + \omega_{j_3} + \omega_j = 0\}.$$

FWI systems and generalizations are expected to provide a good description of a number of physical situations. Examples are pattern formation in vertically oscillated convection [10], multi wave non-linear couplings in elastic structures [8], non-linear optical waves [9], or the four-wave interactions in plasmas [21]. They are also used as a model, cf. [1,5], for the description of gravity-driven surface water waves. Generalizations of four-wave interaction models play a big role in the description of so-called freak waves in deep sea. They are mainly used to disprove the validity of the so-called Rayleigh statistics which predicts a very rare occurrence of freak waves. See [6] for a recent review.

The question, however, if such models really describe the evolution of surface water waves remained open until today. There is a serious problem in order to do so according to the fact that the boundedness of solutions of amplitude $\mathcal{O}(\varepsilon)$ has to be shown on a time interval of length $\mathcal{O}(1/\varepsilon^2)$. A first approach to answer this question positively has been made in [19]. There, for non-linear wave equations with a cubic and a quadratic non-linearity, respectively, it has been explained that the validity proof for the non-linear Schrödinger equation [7,12] can be transfered almost line for line to a validity proof of the systems of the resonant four-wave interaction.

As a next step in answering the validity question for the water wave problem positively we consider here a model for surface water waves on an infinite deep sea as original system, namely

$$\partial_t^2 u = -\omega^2 u + \rho u^2, \tag{1}$$

where $x \in \mathbb{R}^2$, $t \geqslant 0$, and $u(x,t) \in \mathbb{R}$ and where ω and ρ are defined by their symbols in Fourier space

$$\omega^2(k) = |k| \quad \text{and} \quad \rho(k) = |k|/(1+|k|^2)^{1/2}. \tag{2}$$

In the following, we do not distinguish between these operators and their symbols. Hence (1) shares the linear dispersion relation with the water wave problem in case of infinite depth and no surface tension. The non-linear term is chosen to vanish with the same order as in this problem for Fourier wave vectors $k \to 0$. It has chosen to be bounded in order to concentrate on our main purpose and to avoid difficulties with the local existence and uniqueness of solutions of (1). However, the major difficulty in transferring the subsequent ideas to the water wave problem will be due to the local existence and uniqueness of solutions of the water wave problem. For a motivation of (1) with more details see the appendix.

Moreover, this model shares the property with the water wave problem (which is relevant for our purposes) that there are no quadratic resonances. For the problems considered in Ref. [19] there are no quadratic resonances at all, also not for $k_j = 0$. Again the idea of handling the validity question in case of a resonant wave vector $k = 0$ comes from the validity proof for the non-linear Schrödinger equation (NLS) in such a situation [13]. However, the proof has to be modified according to the different degeneracies of ω and ρ for $|k| \to 0$ of our model and the one in [13], representing the water wave problem in case of infinite and finite depth, respectively.

In the following, we restrict ourselves to four resonant wave packets. The system for the resonant four-wave interaction for (1) is then given by

$$\partial_T A_m = -(\nabla_k \omega(k_m)) \cdot \nabla_X A_m + \gamma_m S_m + \sum_{l \in I_4} \gamma_{ml} A_m |A_l|^2 \tag{3}$$

with coefficients $\gamma_m, \gamma_{ml} \in \mathbb{R}$, given in terms of the k_m and ω_m, where

$$S_m = \begin{cases} A_{-2}A_{-3}A_{-4} & \text{if } m = 1, \\ A_{-1}A_{-3}A_{-4} & \text{if } m = 2, \\ A_{-2}A_{-1}A_{-4} & \text{if } m = 3, \\ A_{-2}A_{-3}A_{-1} & \text{if } m = 4, \end{cases}$$

and where we have chosen that branch of ω for which $\omega(k_m) = \omega_m$. Before we state our result we introduce some notations and some basic facts. Fourier transform of a function u is denoted by

$$(\mathscr{F}u)(k) = \hat{u}(k) = \frac{1}{2\pi} \int u(x) e^{-ikx} \, dx.$$