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Immersed boundary method: The existence of approximate solution of the two-dimensional heat equation

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Abstract

This paper deals with the heat equation in which the source term involves a Dirac function and describes the elastic reaction of the immersed boundary. We analyze the existence of the approximate solution in two-dimensional case with Dirac function approximated by differentiable function. We obtain the result via finite element method, the Banach Fixed-Point Theorem and a theorem in nonlinear ordinary differential equations in abstract space.

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Keywords: Immersed boundary method; Finite element method; Nonlinear ordinary differential equations

1. Introduction

The immersed boundary (IB) method was developed for the computer simulation of fluid–structure interaction, especially in biological fluid dynamics (see [8]). The coupling of fluids and solids is the central feature in the study of the mechanics of the heart, arteries, veins, micro-circulation. Nevertheless, the modeling of strong hemodynamic interaction with flexible structure experiencing large deformations and displacements. In particular, the computation requirement with respect to evolving or adaptive meshing becomes considerable in the modeling of deformable cells interacting with viscous fluid and flexible vessels. In the IB method, the structure is thought as a part of the fluid where additional mass may be localized. Therefore, instead of separating the system in its two components coupled by boundary conditions, as is conventionally done, the incompressible Navier–Stokes equations, with a nonuniform mass density and an applied elastic force density, are used in order to describe the coupled motion of the hydro-elastic system in a unified way.

The IB method is at the same time a mathematical formulation and a numerical scheme. The mathematical formulation is based on the use of Eulerian variables to describe the dynamic of fluid and of Lagrangean variables along the moving structure. The force exerted by the structure on the fluid is taken into account by means of a Dirac function constructed according to certain principles. Boffi (see [1]) gave a variational formulation of the problem and provided a suitable

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modification of the IB method which makes use of finite elements. Because the source term in the Navier–Stokes equations involves a Dirac function, the problem is highly nonlinear and presents several difficulties related with the lacking of regularity of the solution of the Navier–Stokes equations due to such source term. Boffi (see [1]) analyzed the existence of the solution in a very simple one-dimensional case. In this paper we deal with the heat equation. We analyze the existence of the approximate solution in two-dimensional case with Dirac function approximated by differentiable function. We obtain the result via finite element method, the Banach Fixed-Point Theorem and a theorem in nonlinear ordinary differential equations in abstract space.

2. Problem

2.1. Notation

Let Ω denote the square domain in R^2. We shall be concerned with the spaces of two-dimensional vector functions. We shall use the notations

$$\mathbf{C}[a,b]=\{C[a,b]\}^2,\quad \mathbf{H}^m(\Omega)=\{H^m(\Omega)\}^2,\quad \mathbf{H}_0^1(\Omega)=\{H_0^1(\Omega)\}^2,\quad \mathbf{L}^2(\Omega)=\{L^2(\Omega)\}^2,$$

and we suppose that these product spaces are equipped with the usual product norm. The norm on $\mathbf{C}[a,b]$ is denoted by $\|\cdot\|$. The norm on $\mathbf{L}^2(\Omega)$ is denoted by $\|\cdot\|_0$. The norm on $\mathbf{H}^m(\Omega)$ is denoted by $\|\cdot\|_m$. $(\cdot,\cdot)$ stands for the scalar product on $\mathbf{L}^2(\Omega)$, while $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $\mathbf{H}^{-1}(\Omega)$ and $\mathbf{H}_0^1(\Omega)$.

$\mathscr{L}([c,d];\mathbf{C}[a,b])$ denotes the space of continuous functions from $[c,d]$ into $\mathbf{C}[a,b]$ and is equipped with the Banach norm

$$\|\mathbf{x}\|_c=\sup_{c\leqslant t\leqslant d}\max_{a\leqslant s\leqslant b}|\mathbf{x}(s,t)|.$$

In this paper, we always suppose that $\widetilde{T}>0$ is a given constant, and constant $T\in(0,\widetilde{T}]$. Let

$$G=\mathscr{L}([0,T];\mathbf{C}[0,L]),\quad E=\{\mathbf{x}\in\mathbf{C}[0,L]:\mathbf{x}(0)=\mathbf{x}(L)\}.$$

For $\mathbf{x}\in G$, $\mathbf{x}(\cdot,t)\in\mathbf{C}[0,L]$, $\forall t\in[0,T]$, $\mathbf{x}(\cdot,t)$ is simply denoted by $\mathbf{x}(t)$.

Given $\mathbf{X}_0\in E$, let

$$\mathbb{X}=\{\mathbf{X}\in\mathscr{L}([0,T];E):\mathbf{X}(0)=\mathbf{X}_0\}.$$

Then $\mathbb{X}$ is a closed set in $\mathscr{L}([0,T];E)$.

Given $\delta_h:R^2\to R^2$, $\mathbf{f}:\mathbf{C}([0,L]\times[0,\widetilde{T}])\to R^2$, we first present the following Assumption A:

$$\delta_h\in\mathbf{H}_0^2(R^2),\quad \mathbf{f}\in\mathbf{C}([0,L]\times[0,\widetilde{T}]).$$

2.2. Formulation

The authors (see [1,4]) considered the model problem of a viscous incompressible fluid in Ω containing an immersed massless elastic boundary in the form of a curve. To be more precise, for all $t\in[0,T]$, let Γ_t be a simple curve, the configuration of which is given in a parametric form, $\mathbf{X}(\mathbf{s},\mathbf{t})$, $0\leqslant s\leqslant L$, $\mathbf{X}(0,t)=\mathbf{X}(L,t)$. The equations of motion of the system are

$$\rho\frac{\partial\mathbf{u}}{\partial t}-\mu\Delta\mathbf{u}+\mathbf{u}\cdot\Delta\mathbf{u}+\Delta p=\mathbf{F}\quad\text{in }\Omega\times(0,T), \tag{1}$$

$$\varDelta\cdot\mathbf{u}=0\quad\text{in }\Omega\times(0,T), \tag{2}$$

$$\mathbf{F}(x,t)=\int_0^L\mathbf{f}(s,t)\delta(x-\mathbf{X}(s,t))\,\mathrm{d}s\quad\forall x\in\Omega,\ \ t\in(0,T), \tag{3}$$

$$\frac{\partial\mathbf{X}}{\partial t}(s,t)=\mathbf{u}(\mathbf{X}(s,t),t)\quad\forall s\in[0,L],\ \ t\in(0,T). \tag{4}$$