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Direction of Hopf bifurcation in a delayed Lotka–Volterra competition diffusion system

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This paper is concerned with a delayed Lotka–Volterra two species competition diffusion system with a single discrete delay and subject to homogeneous Dirichlet boundary conditions. The main purpose is to investigate the direction of Hopf bifurcation resulting from the increase of delay. By applying the implicit function theorem, it is shown that the system under consideration can undergo a supercritical Hopf bifurcation near the spatially inhomogeneous positive stationary solution when the delay crosses through a sequence of critical values.

1. Introduction

In recent years, the study on population models in various fields of mathematical biology has become a very popular topic since the pioneering theoretical works by Lotka [7] and Volterra [15] in which the well-known Lotka–Volterra two species predator-prey system described by the ordinary differential equations was proposed and studied. Thereafter, based on the classical Lotka–Volterra two species predator-prey model, many population models reflecting various interactions between two species have been proposed. For example, when there are two different species in a certain environment and they live by competing the resources, if the past history has the effect on the dynamical behaviors of system and the species will migrate towards regions of lower population density to add the possibility of survival, under the assumption that in the absence of one of the species, the growth of the other species will be governed by the delayed logistic diffusion equation and the dimension of space will be taken as one, then in this case the dynamics of growth of two species should be described by the following delayed Lotka–Volterra competition diffusion system

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = d_1 \dfrac{\partial^2 u(t,x)}{\partial x^2} + u(t,x)[r_1 - a_{11}u(t-\tau_{11},x) - a_{12}v(t-\tau_{12},x)], \\ \dfrac{\partial v(t,x)}{\partial t} = d_2 \dfrac{\partial^2 v(t,x)}{\partial x^2} + v(t,x)[r_2 - a_{21}u(t-\tau_{21},x) - a_{22}v(t-\tau_{22},x)], \end{cases} \tag{1.1}$$

where $u(t,x)$ and $v(t,x)$ represent the densities of two species at time t and space x, respectively; $d_1 > 0$ and $d_2 > 0$, $r_1 > 0$ and $r_2 > 0$, $\tau_{11} \geq 0$ and $\tau_{22} \geq 0$ denote the diffusion coefficients, intrinsic growth rates, gestation periods of species u and v, respectively; $\tau_{12} \geq 0$ and $\tau_{21} \geq 0$ denote the maturity time for species v and u, respectively; a_{ij} $(i,j = 1,2)$ are all positive constants.

System (1.1) under various circumstances has been investigated extensively by many researchers and many interesting results have been obtained, see [2–5,11–14,17,18]. For example, it is shown in the literature [2,17] that all the trajectories

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in the first quadrant will converge to one of the nonnegative equilibria when there are no diffusion effects and delays in system (1.1), that is, $d_1 = d_2 = 0$ and $\tau_{ij} = 0$ $(i, j = 1, 2)$. In addition, Sprott, Wildenberg and Azizi [11] as well as Sprott, Vano and Wildenberg [12] considered the spatiotemporal chaotic dynamics of system (1.1) in the case when there are no diffusion effects and delays, and (1.1) represents a circular competition system. However, when there are no diffusion effects but there are delays, Tang and Zou [13,14] investigated the global attractivity of system (1.1). In references [3,4], Kuang and Smith investigated the global stability of the delayed Lotka–Volterra type diffusion systems subject to Neumann boundary conditions. When the spatial region is taken as the whole real line, Li, Lin and Ruan [5] investigated the existence of traveling wave solutions of system (1.1).

If the environment is limited and surrounded by a totally unfavorable region in which the population density cannot attain positive values, then according to the knowledge in [16] the homogeneous Dirichlet boundary and initial value conditions

$$\begin{cases} u(t,x) = v(t,x) = 0, \quad x = 0, \pi, t \geq 0, \\ u(t,x) = \phi(t,x) \geq 0, \qquad v(t,x) = \phi(t,x) \geq 0, \quad (t,x) \in [-\tau, 0] \times [0, \pi] \end{cases} \tag{1.2}$$

should be imposed on system (1.1), where

$$\phi, \psi \in C = C([-\tau, 0], H_0^1[0, \pi]) \quad \text{and} \quad \tau = \max_{1 \leq i,j \leq 2} \{\tau_{ij}\}.$$

It is well known that the study on dynamics of systems under Dirichlet boundary conditions is more difficult because nontrivial equilibrium or periodic solution (if it exists) is spatially nonconstant. Therefore, the study on the dynamics of system (1.1) with conditions (1.2) is very rare. Recently, Lin and Pedersen [6], Ruan [9], and Ryu and Ahn [10] studied the existence of positive stationary solutions of competition diffusion system with cross-diffusion and Dirichlet boundary conditions. It can be seen easily from these literature that system (1.1) with condition (1.2) has only a trivial stationary solution when $r_i \leq d_i$ $(i = 1, 2)$ and has at least a positive stationary solution when $r_i > d_i$ $(i = 1, 2)$ and the other conditions are satisfied.

In this paper, we consider mainly the stability of positive stationary solution and the direction of Hopf bifurcation resulting from the increase of delay for system (1.1) with condition (1.2) under the assumption that $d_1 = d_2 = d, r_1 = r_2 = r$ and $\tau_{ij} = \tau$ $(i, j = 1, 2)$. Make the change of variables and parameters $\bar{t} = dt,\ \bar{\tau} = d\tau, \bar{a}_{ij} = \frac{a_{ij}}{r}$, and drop the bars for the simplicity of notations and let $k = \frac{d}{r}$. Then system (1.1) with conditions (1.2) is equivalent to the following system

$$\begin{cases} u_t = u_{xx} + ku(t,x)[1 - a_{11}u(t-\tau,x) - a_{12}v(t-\tau,x)], \\ v_t = v_{xx} + kv(t,x)[1 - a_{21}u(t-\tau,x) - a_{22}v(t-\tau,x)], \\ t > 0, \quad 0 < x < \pi, \\ u(t,x) = v(t,x) = 0, \quad x = 0, \pi, t \geq 0, \\ u(t,x) = \phi(t,x), \qquad v(t,x) = \phi(t,x), \quad (t,x) \in [-\tau, 0] \times [0, \pi]. \end{cases} \tag{1.3}$$

Under the condition that

(H) $\frac{a_{11}}{a_{21}} > 1 > \frac{a_{12}}{a_{22}}$,

it is shown in the literature [18] that system (1.3) can bifurcate a unique spatially inhomogeneous positive stationary solution from trivial solution $(0, 0)$ if $0 < k - 1 \ll 1$ and this positive stationary solution is asymptotically stable if the delay τ is less than a certain critical value. In addition, system (1.3) can also undergo a spatially inhomogeneous Hopf bifurcation near the positive stationary solution when the delay τ crosses through a sequence of critical values. However, under the condition (H), the direction of Hopf bifurcation is not investigated. Therefore, in this paper, we study mainly the direction of Hopf bifurcations resulting from the increase of delay only when the condition (H) is satisfied. It is shown that, under the assumption (H), system (1.3) can undergo a spatially inhomogeneous *supercritical* Hopf bifurcation near the positive stationary solution when the delay τ crosses through a sequence of critical values.

The remaining parts of this paper are organized as follows. In Section 2, we will list the existence, uniqueness result and asymptotic expressions of positive stationary solution of system (1.3) under the condition (H) according to [1,18]. In Section 3, the stability of positive stationary solution of system (1.3) and the existence of Hopf bifurcation near positive stationary solution are also stated. In Section 4, by applying the implicit function theorem, we investigate the direction of Hopf bifurcations obtained in Section 3 and find that system (1.3) can undergo a *supercritical* Hopf bifurcation near positive stationary solution when the delay crosses through a sequence of values.

2. Positive stationary solution

In this section, we shall give the existence, uniqueness and asymptotic expressions of stationary solution of system (1.3) when $0 < k - 1 \ll 1$ according to the results of [18].