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Existence of a global weak solution for a 2D viscous bi-layer Shallow Water model

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ABSTRACT

We consider a non-linear viscous bi-layer shallow water model with capillarity effects and extra friction terms in a two-dimensional space. This system is issued from a derivation of three-dimensional Navier–Stokes equations with a water-depth depending on friction coefficients. We prove an existence result for a global weak solution in a periodic domain $\Omega = \mathbb{T}^2$.

1. Introduction

Shallow-Water flows cover a very large number of geophysical and engineering applications, such as ocean circulation, coastal areas, rivers, lakes, avalanches, ... But, in many situations one layer of Shallow-Water cannot be used to model the system. The simplest example is the flow in the Strait of Gibraltar. It is necessary in this case to consider two layers of water. Indeed, the conservation of the volume of water and salinity in the basin indicates the presence of two opposite flows: the surface Atlantic water and the deeper, denser Mediterranean water flowing into the Atlantic. Thus, it is necessary to consider at least a two layer model if we want to simulate the flow in this region (cf. [8]). We assume that for these phenomena one can make an appropriate Shallow-Water approximation. For this purpose, we can find many derivations of bi-layer and multi-layer Shallow-Water models. In [1], Audusse derived a multi-layer Shallow-Water model to extend the case of one layer established by Gerbeau and Perthame in [15]. In this work, using the hydrostatic pressure and the kinematic boundary conditions, they derived momentum equations of the form:

$$\partial_t \int_{H_{\alpha-1}}^{H_\alpha} u\mathrm{d}z + \partial_x \int_{H_{\alpha-1}}^{H_\alpha} u^2\mathrm{d}z + gh_\alpha\partial_x h = \frac{\nu_0}{\epsilon}\partial_z u(H_\alpha(t,x)) - \frac{\nu_0}{\epsilon}\partial_z u(H_{\alpha-1}(t,x))$$

and use at the leading order a finite difference method with respect to the vertical variable when the equation is an interface equation to deduce the friction term:

$$\mu\partial_z u(H_\alpha) = \mu\frac{U_{\alpha+1} - U_\alpha}{h_{\alpha+1} + h_\alpha}.$$

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In [22], Peybernes deduce a bi-layer viscous Shallow-Water model which takes into account the friction at the interface. But, instead of asymptotic analysis several assumptions of simplifications are used in the boundary conditions to deduce the final system. Also, the energy of the system is obtained under a restrictive hypothesis on the data.

On the other hand, we propose in this paper a new viscous bi-layer Shallow-Water model with different constant densities. Following the work performed in [15] for one layer in one dimensional case and in [17] for one layer but in the two dimensional case, here the considered model is a simplified system of a general one obtained in [14]. In [17], a viscous one layer of two dimensional Shallow-Water system is derived by Marche. The originality in this work is the introduction of surface-tension term through the capillary effects at the free surface and quadratic friction term at the bottom. Such surface-tension and quadratic friction terms have been useful to establish the existence of global weak solutions in [2]. Our model also takes into account a friction term on the bottom and a capillary term on the interface and on the free surface. Another work related to the derivation of a 2D Shallow-Water model has been done by Ferrari and Saleri in [13]. In particular, the authors include the atmospheric pressure in the derivation. For the sake of brevity, we have not included in this work the deduction of our new viscous bi-layer model, see [14] for detail.

We prove the existence of a global weak solution for the considered system. The analysis developed here is based on the techniques used by Bresch, Desjardins and Lin in [2,3] and [6]. In these works, they obtain the existence of a global weak solution for a 2D Shallow Water system and a Korteweg system with a diffusion term of type $\nu\,\mathrm{div}(hD(u))$. They prove that the considered systems is energetically consistent without any restriction on the data. The key point of this proof is based in a estimate of a new entropy (in mathematical sense), called "mathematical BD entropy", which gives a bound of the term $\nabla\sqrt{h}$. This inequality is extended later to a more general Navier–Stokes equation with an algebraic relation between the shear and the bulk viscosity coefficients. But, the authors used quadratic friction terms and capillary effects to get the stability of the system in [2]. More recently, another proof also based on the "BD entropy" estimate of the stability for the Navier–Stokes equations for barotropic compressible fluids is developed in [18] by Mellet and Vasseur. Note that this analysis includes the case of Shallow-Water without any regularizing term. Their analysis is based on the estimate of ρu^2 which is enough to get a compactness result. In fact, this estimate replaces that of $h^{1/3}u$ in [2] obtained by using a drag term of the form $r|h||u|u$. But, it is not actually possible to construct a suitable approximate sequences of weak solutions with this method.

In [12,22], the authors prove the existence of a global weak solution of a bi-layer Shallow-Water model without any friction term but with a diffusion term of the form $\nu\Delta u$. This analysis uses the method developed by Orenga in [21] and the system is energetically consistent only for small enough initial data. Others works concerning the existence of a global weak solution of a bi-layer Shallow-Water using the preceding method can also be found in [9,19,20].

In this work, we consider in a periodic domain Ω, a system composed by two layers of immiscible fluids with different and constant densities (ρ_1 and ρ_2, resp.) and viscosities (ν_1 and ν_2, resp.).

From now on, index 1 refers to the deeper layer and index 2 to the upper layer of the flow. So, h_i, u_i for $i = 1, 2$ denote the thickness and the velocity field of each layer. We define h to be $h = h_1 + h_2$. We assume that the friction coefficient at the bottom c_0 and the coefficients α_1, α_2 representing respectively the interface and free surface tensions coefficients are positive.

The model proposed here reads as:

$$\partial_t h_1 + \mathrm{div}(h_1 v_1) = 0; \tag{1}$$

$$\begin{aligned}&\rho_1\partial_t(h_1v_1) + \rho_1\,\mathrm{div}(h_1v_1\otimes v_1) - 2\nu_1\,\mathrm{div}(h_1D(v_1)) + \rho_1 g h_1\nabla h_1 + \rho_2 g h_1 \nabla h_2\\ &\quad - \left(1+\frac{c_0\beta(h_1)h_1}{6\nu_1}\right)\mathrm{fric}(v_1, v_2) + c_0\beta(h_1)v_1 - \alpha_1 h_1\nabla(\Delta h_1) - \alpha_2 h_1 \nabla(\Delta h_2) = 0;\end{aligned} \tag{2}$$

$$\partial_t h_2 + \mathrm{div}(h_2 v_2) = 0; \tag{3}$$

$$\begin{aligned}&\rho_2\partial_t(h_2v_2) + \rho_2\,\mathrm{div}(h_2v_2\otimes v_2) - 2\nu_2\,\mathrm{div}(h_2D(v_2))\\ &\quad + \rho_2 g h_2\nabla h_2 + \rho_2 g h_2\nabla h_1 + \mathrm{fric}(v_1, v_2) - \alpha_2 h_2\nabla(\Delta h) = 0\end{aligned} \tag{4}$$

with initial conditions:

$$h_{i|t=0} = h_{i_0} \geq 0, \qquad h_i v_{i|t=0} = m_{i_0}, \tag{5}$$

for which we assume the following regularity:

$$\begin{aligned}&h_{i_0} \in L^2(\Omega), \qquad \nabla h_{i_0} \in (L^2(\Omega))^2, \qquad \nabla\sqrt{h_{i_0}} \in (L^2(\Omega))^2\\ &\frac{|m_{i_0}|^2}{h_{i_0}} \in L^1(\Omega), \qquad \log_-(h_{i_0}) \in L^1(\Omega).\end{aligned} \tag{6}$$

The function β depending on h_1 is one of the drag coefficients given by:

$$\beta(h_1) = \left(1 + \frac{c_0}{3\nu_1}h_1\right)^{-1}. \tag{7}$$

We denote by $D(v)$ the strain tensor, defined by $D(v) = \frac{\nabla v + \nabla^t v}{2}$, and by $A(v)$, the vorticity tensor such that $A(v) = \frac{\nabla v - \nabla^t v}{2}$.