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An equivalent definition of packing dimension and its application

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Abstract

An equivalent definition of packing dimension is given for a set in d-dimensional Euclidean space by using its component sets as packings. It is applied to determine the packing dimensions of a class of subsets with prescribed relative group frequencies.

Keywords: Moran set; Component set; Packing dimension; J-type packing

1. Introduction

The packing dimension of a set in d-dimensional Euclidean space plays an important role in fractal geometry. We first recall its definition. For a set $E \subseteq \mathbb{R}^d$ and $\delta > 0$, a δ-packing of E is defined as a collection of at most countable number of disjoint balls of radii at most δ with centers in E. Fixing $s \geq 0$, the s-dimensional packing measure of E is defined by

$$\mathcal{P}^s(E) = \inf \left\{ \sum_i \mathcal{P}_0^s(E_i) : E \subseteq \cup_{i=1}^{\infty} E_i \right\},$$

where $\mathcal{P}_0^s(E_i) = \lim_{\delta \to 0+} \sup \sum_j |B_j|^s$ and the sup is taken over all δ-packing $\{B_j\}$ of E_i. The packing dimension of E is then given by

$$\dim_P E = \sup\{s : \mathcal{P}^s(E) = \infty\} = \inf\{s : \mathcal{P}^s(E) = 0\}.$$

Since the packings used for $\mathcal{P}_0^s(E_i)$ are restricted to be disjoint balls centered at E_i, this often makes inconvenient for evaluating the packing dimension of E. Thus, it is wise and necessary to find out an alternative way to relax the restriction on a packing but induce the same value for the packing dimension as that done above. Some results on this topic were given by Taylor and Tricot in [7]. Since the packing dimension is σ-stable, i.e., $\dim_P \bigcup_{i=1}^{\infty} E_i = \sup_i \dim_P E_i$, we may assume that E is bounded. In the present paper, we confine ourselves to the so-called Moran sets defined below as in (1). As one can see, a Moran set is a well-constructed compact set which includes the self-similar sets satisfying open set condition and even any closed cubes. Thus each bounded set E can always be considered as a subset of a Moran set, say F. What we like to do is to make use of the component sets of F as packings for evaluating the packing dimension of E. A Moran set is defined as follows.

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Denote $\Omega = \{1, 2, \ldots, r\}$, where $r \geq 2$. Let $\Omega^* = \bigcup_{m=0}^{\infty} \Omega^m$ with $\Omega^m = \{\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(m)) : \sigma(j) \in \Omega\}$ for $m \in \mathbb{N} \cup \{0\}$ (Ω^0 consists of the empty word $\emptyset$) and $\Omega^{\mathbb{N}} = \{\sigma = (\sigma(1), \sigma(2), \ldots) : \sigma(j) \in \Omega\}$. $|\sigma|$ is used to denote the length of word $\sigma \in \Omega^*$. For $\sigma \in \Omega^*$ and $\tau \in \Omega^* \cup \Omega^{\mathbb{N}}$, $\sigma * \tau$ denotes their concatenation. In particular, $\emptyset * \tau = \tau$. For $\sigma \in \Omega^m$, let $C(\sigma) = \{\tau \in \Omega^{\mathbb{N}} : \tau|m = \sigma\}$, called the cylinder set with base σ, where $\tau|m = (\tau(1), \tau(2), \ldots, \tau(m))$.

Fix a constant $0 < c < 1$ and positive real numbers $0 < a_i < 1$, $i = 1, 2, \ldots, r$. Assume that a collection $(J_\sigma)_{\sigma \in \Omega^*}$ of compact sets in $\mathbb{R}^d$ has the following features:

[A1] nested property: for $\sigma \in \Omega^*$ and $i \in \Omega$, $J_{\sigma * i} \subseteq J_\sigma$;
[A2] nonoverlapping property: all J_σs with $\sigma \in \Omega^m$ are pairwise nonoverlapping in the sense that $J_\sigma \cap J_\tau$ is of zero d-dimensional Lebesgue measure for any distinct $\sigma, \tau \in \Omega^m$;
[A3] regular sizes for J_σs: for $\sigma \in \Omega^*$ and $i \in \Omega$, $|J_{\sigma * i}| = a_i |J_\sigma| > 0$, where, if no confusion occurs, $|J_\sigma|$ denotes the diameter of J_σ;
[A4] regular sizes for the interior of J_σs: each $J_\sigma, \sigma \in \Omega^*$ contains an open ball with diameter $c|J_\sigma|$.

The *Moran set* F associated with $(J_\sigma)_{\sigma \in \Omega^*}$ is defined as the nonempty compact set

$$F = \bigcap_{m=0}^{\infty} \bigcup_{\sigma \in \Omega^m} J_\sigma. \tag{1}$$

The definition of Moran sets here is close to that in [1,6] with a bit variation, but simpler than that in [9] where a more general structure is discussed (Readers can refer to [9] and more references therein for related results on this kind of structure). Obviously, self-similar sets satisfying open set condition are Moran sets. The latter lose the property of similitude, but keep some typical properties exhibited by the former, e.g., they can be encoded by elements from $\Omega^{\mathbb{N}}$ and

$$\dim_H F = \overline{\dim}_B F = s, \ 0 < \mathcal{H}^s(F) \leq \mathcal{P}^s(F) < \infty,$$

where s is determined by $\sum_{j=1}^{r} a_j^s = 1$. The compact sets $J_\sigma, \sigma \in \Omega^*$ are generally referred to as *component sets* of F. In particular, J_σ is referred to as an mth-level component set of F if $\sigma \in \Omega^m$. Define ϕ: $\Omega^{\mathbb{N}} \longrightarrow \mathbb{R}^d$ by

$$\{\phi(\sigma)\} = \bigcap_{m=0}^{\infty} J_{\sigma|m}. \tag{2}$$

It is easy to see that $\phi(\Omega^{\mathbb{N}}) = F$ and $\phi(C(\sigma)) = F \bigcap J_\sigma$ by (2). But ϕ may not be injective. Let ρ be the metric on $\Omega^{\mathbb{N}}$ such that for any $\sigma, \tau \in \Omega^{\mathbb{N}}$

$$\rho(\sigma, \tau) = 2^{-\min\{i:\, \sigma(i) \neq \tau(i)\}},$$

with the convention $\rho(\sigma, \sigma) = 0$. Let F be equipped with the Euclidean metric. Then ϕ is continuous. Thus each $x \in F$ can be encoded via ϕ: a sequence $\sigma \in \Omega^{\mathbb{N}}$ is called a location code of $x \in F$ if $\phi(\sigma) = x$. Therefore, ϕ is also called the coding map and $\Omega^{\mathbb{N}}$ is called the code space (or symbolic space). As a result, F is a projection of $\Omega^{\mathbb{N}}$ on $\mathbb{R}^d$ via ϕ.

Let $0 < R < a_{\min}|J_\emptyset|$ where $a_{\min} := \min_{1 \leq i \leq r} a_i$. A component set J_σ of F is termed as an R-size component set if

$$|J_\sigma| \leq R \quad \text{and} \quad |J_{\sigma|(|\sigma|-1)}| > R.$$

It is easy to see that for any $0 < R < a_{\min}|J_\emptyset|$, the set of all R-size component sets of F is a nonoverlapping finite R-covering of F. Hence, by means of Lemma 9.2 in [3], the requirement (see [A4]) that J_σ contains an open ball of diameter $c|J_\sigma|$ implies an important fact: there exists a positive integer ϑ, independent of R and $x \in \mathbb{R}^d$, such that any ball $B_R(x)$ with radius R and center at x intersects at most ϑ of the R-size component sets of F. Many analogues of this fact appear in this paper. A direct sequel leads to an important property of ϕ:

$$\sup_{x \in F} \#\{\phi^{-1}(x)\} < \vartheta,$$