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Longitudinal normals and the existence of acoustic axes in crystals

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Abstract

We obtain three conditions on the phase speeds of the longitudinal and transverse waves propagating along the longitudinal normals in a crystal so that each of these conditions guarantees existence of acoustic axes in this crystal. The result is based on the properties of the rational-valued topological degree and of the index of a singular point for some classes of discontinuous mappings. In addition, we give an upper estimate of the number of acoustic axes in a crystal and show some interrelation between their indices.

Keywords: Topological degree; Acoustic axis; Longitudinal normal; Crystal; Index of an acoustic axis

0. Introduction

A crystal is an anisotropic elastic medium. Three different elastic waves may simultaneously propagate in any direction within it. The displacement vectors of its atoms (polarization vectors) for these waves are mutually orthogonal. The phase speeds of these waves may coincide for some directions in a crystal. Such directions are called acoustic axes. On the other hand, for some directions in a crystal one of these waves is longitudinal, i.e. the polarization vectors are parallel to the propagation direction, and the two others are transverse, i.e. the polarization vectors are orthogonal to the propagation direction. The directions with this property are called longitudinal normals. Acoustic axes and longitudinal normals are specific propagation directions, and they play an important role in the study of wave motion in crystals (see [5, Section 17]).

Acoustic axes and longitudinal normals (and the properties of the elastic waves propagating along and near the specific directions) have attracted the attention of many investigators (see, for instance, [1–5,8,12]). Various conditions under which a direction in a crystal is an acoustic axis were deduced in [8,12].

The application of topological methods turned out to be rather effective in this area of research. For acoustic axes, the topological arguments were first used by Alshits, Sarychev and Shuvalov (see [1]). They introduced the concept of index of an acoustic axis on the basis of the concept of the index of a singular point of the polarization vector field around the acoustic axis. In turn, the index of a singular point of a vector field in [1] is defined by analogy with the classical case, where the index of a singular point of a vector field is the algebraic number of revolutions of the field

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around a singular point (see [9 (Subsections 1.3 and 3.3),13 (Section 6)]). However in many situations considered in [1] the vector field made only a half-turn clockwise or counterclockwise, and by analogy with the classical case the authors let the index equal either to $+\frac{1}{2}$ (if the sense of rotation coincides with the direction of turn of the vector field) or $-\frac{1}{2}$ (in the opposite case). However, due to the discontinuity of the arising vector fields, this definition was more intuitive than strictly mathematical. Nevertheless, with the help of the introduced concept of the index, the authors of [1] give a classification of acoustic axes and analyze the stability of acoustic axes and the local structure of polarization vector fields.

The application of topological methods for the study of longitudinal normals is also very fruitful (see [2]). In particular, it was shown that a crystal has at least three longitudinal normals.

In [14], the acoustic axes were investigated under the (rather customary) assumption that the phase speed of one of the three waves mentioned above (the one for which the polarization vector makes the smallest acute angle with the propagation direction) is greater than the speeds of the two other waves for each propagation direction. In this case, a strict mathematical definition of the index of an acoustic axis was given, and the existence of at least one acoustic axis was proved [14, Corollary 4.1].

However, it seems to be more natural to study the acoustic axes without the assumption on the phase speeds which has just been described, as it was done, for instance, in [4,12]. On the other hand, acoustic axes do not exist *a priori* (see Remark 4.2), so one has to find some other conditions which ensure the existence of acoustic axes. Three such conditions are given in Theorem 4.1 of the current paper. They concern only the phase speeds of the longitudinal and transverse waves propagating along the longitudinal normals in a crystal (the number of which usually does not exceed thirteen [2]), in contrast to the hypothesis mentioned in the previous paragraph, which refers to all propagation directions. Then we obtain some information concerning the interrelationships between the indices of acoustic axes (Theorem 4.2), and show that the number of acoustic axes does not exceed sixteen under some (not highly restrictive) condition on the parameters of a crystal (Theorem 4.3).

The analysis of the polarization vector fields (which arose in [1]) shows that they are not continuous, but have some additional properties, in particular, the superpositions of maps, corresponding to these vector fields, with some fixed maps are continuous. For these superpositions, it is possible to introduce the topological degree and to give a strict mathematical definition of the degree for a certain class of discontinuous maps on this basis. This enables us to introduce a concept of an index of singular points for the corresponding vector fields (and therefore a concept of the index of an acoustic axis). It turns out that the degree and the index defined in this way may attain any rational values (although for continuous maps they are integers and coincide with the usual conceptions of degree and index). In addition, the classical Euler's theorem on the sum of indices of singular points of a vector field on a sphere can be generalized to the case of discontinuous fields with singularities of the rational index (Theorem 3.1). These issues are described in Sections 1–3, and the application to the study of acoustic axes is carried out in Section 4.

0.1. Notations

In this paper, open bounded subsets of $\mathbb{R}^n$, $n > 1$, are called *domains*. The closure of any domain $D \subset \mathbb{R}^n$ is denoted by the symbol $\overline{D}$, and the boundary is denoted by ∂D. The notation $C(\overline{D})$ stands for the Banach space of continuous functions on $\overline{D}$ with values in $\mathbb{R}^n$.

The imaginary unit is denoted by the symbol $\imath$.

Let $\langle\cdot,\cdot\rangle$ and $\times$ denote the scalar and vector product in $\mathbb{R}^3$, respectively. Let S^2 stand for the unit sphere $\{x \in \mathbb{R}^3 | \sqrt{\langle x, x\rangle} = \|x\| = 1\}$. Let $U(a)$ denote the hemisphere centred at the point a, i.e. the set $\{x \in S^2 | \langle x, a\rangle > 0\}$. The symbol $\mathbb{D}$ will stand for the open unit disk in $\mathbb{R}^2$.

For any vectors a and b in S^2 such that $\langle a, b\rangle = 0$, we introduce the map

$$\Pi_a^b : \mathbb{R}^3 \to \mathbb{R}^2,$$

$$\Pi_a^b(x) = \begin{pmatrix} \langle x, b\rangle \\ \langle x, a \times b\rangle \end{pmatrix}. \tag{0.1}$$

Note that $\Pi_a^b|_{\overline{U}(a)}$ is a homeomorphism from $\overline{U}(a)$ onto $\overline{\mathbb{D}}$.

If $\varphi : S^2 \to \mathbb{R}^3$ is a *tangential vector field*, i.e.

$$\langle\varphi(b), b\rangle = 0 \quad \forall b \in S^2, \tag{0.2}$$