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ScienceDirect
Nonlinear Analysis: Real World Applications 10 (2009) 1210–1219
Nonlinear Analysis
Real World Applications
www.elsevier.com/locate/nonrwa

The finite element analysis of a fractional-step method for the time-dependent linear elasticity equations☆

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Received 6 January 2007; accepted 8 January 2008

Abstract

This paper provides a convergence analysis of a fractional-step method for time-dependent system of equations of linear elasticity by means of finite element approximations. Error estimates in finite time are given. And it is verified that provided the time-step τ is sufficiently small, the proposed algorithm yields for finite time T an error of $O(h^2+\tau)$ in the L^2-norm for the displacement field $\boldsymbol{u}$ and an error estimate of $O(h+\tau)$ in the H^1-norm, where h is the mesh size. In addition, under stronger initial conditions we obtain an error estimate of $O(h+\tau)$ in the L^2-norm for the divergence field ϕ.

Keywords: Linear elasticity equations; Fractional-step method; Finite element approximation; Error analysis

1. Introduction

We consider the following time-dependent system of equations of linear elasticity

$$\begin{cases} \dfrac{\partial^2 \boldsymbol{u}}{\partial t^2} - \mu\triangle\boldsymbol{u} - (\lambda+\mu)\,\mathbf{grad}(\operatorname{div}\boldsymbol{u}) = \boldsymbol{f}, \quad \text{in } \Omega\times[0,T], \\ \boldsymbol{u} = \mathbf{0}, \quad \text{on } \partial\Omega\times[0,T], \\ \boldsymbol{u}(0) = \boldsymbol{u}_0, \qquad \boldsymbol{u}_t(0) = \boldsymbol{u}_1, \quad \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded domain in $R^d (d = 2 \text{ or } 3)$ with a sufficiently smooth boundary $\Gamma = \partial\Omega$. The unknown $\boldsymbol{u}$ denotes the displacement vector, $\boldsymbol{f} \in L^2(\Omega)^d$ a given body force and, λ and μ are the so-called lamé constants.

☆ The project was supported by the National Natural Science Foundation of China (Grant No. 10661005), the Guangxi Science Foundation (Grant No. 0640165), and NTU start-up funding (Grant No. M58110011).
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1468-1218/$ - see front matter
doi:10.1016/j.nonrwa.2008.01.002

It is easy to see that (1.1) can be rewritten in the following form

$$\begin{cases} \dfrac{\partial^2 \boldsymbol{u}}{\partial t^2} - \mu \triangle \boldsymbol{u} + \nabla \phi = \boldsymbol{f}, \quad \text{in } \Omega \times [0, T], \\ \nabla \cdot \boldsymbol{u} = -\eta \phi, \quad \text{in } \Omega \times [0, T], \\ \boldsymbol{u} = \boldsymbol{0}, \quad \text{on } \partial\Omega \times [0, T], \\ \boldsymbol{u}(\cdot, 0) = \boldsymbol{u}_0, \qquad \boldsymbol{u}_t(\cdot, 0) = \boldsymbol{u}_1, \quad \text{in } \Omega, \end{cases} \tag{1.2}$$

where $\eta = 1/(\lambda + \mu)$. Without loss of generality we assume that $\eta \in [0, \bar{\eta}]$, where $\bar{\eta}$ is some finite constant.

As is known to all, the fractional-step projection methods of Chorin [1,2] and Shen [3,4] have been successfully applied to solve the incompressible Navier–Stokes equations in primitive variables in recent years. In the method, the convection-diffusion and the incompressibility are dealt within two different substeps and therefore the original problem is converted into solving a convection-diffusion problem and a Poisson problem at each step. Thus, the projection methods have the advantage of needing less computation in comparison with the coupled techniques such as those that are based on the Uzawa operator [8–11,13,16].

In view of the above-mentioned virtues, the aim of this paper is to present and analyze a fractional-step algorithm for the time-dependent system of equations of linear elasticity by means of finite element approximations. It is shown that provided the time-step τ is sufficiently small, the proposed algorithm yields for finite time T an error of $O(h^2+\tau)$ in the L^2-norm for the displacement $\boldsymbol{u}$ and an error of $O(h + \tau)$ in the H^1-norm(or the L^2-norm for the divergence field ϕ under stronger initial-value assumptions).

This paper is organized as follows. In Section 2, we introduce some preliminaries. The fractional-step algorithm is proposed in Section 3. Section 4 is devoted to the error estimates with mild regularity assumptions on the solution of the continuous problems. An additional error estimates on the unknown ϕ is carried out in Section 5 with stronger initial-value assumptions.

2. Preliminaries

First let us introduce some notations and hypotheses. As usual, $W^{s,p}(\Omega)$ denotes the real Sobolev space, $0 \le s < \infty, 0 \le p \le \infty$, equipped with the norm $\|\cdot\|_{s,p}$ and semi-norm $|\cdot|_{s,p}$. The space $W_0^{s,p}$ is the completion of the space of smooth functions compactly supported in Ω with respect to the $\|\cdot\|_{s,p}$ norm. For $p = 2$, we denote the Hilbert spaces $W^{s,2}(\Omega)$ (resp., $W_0^{s,2}(\Omega)$) by $H^s(\Omega)$ (resp., $H_0^s(\Omega)$). The related norm is denoted by $\|\cdot\|_s$. The dual space of $H_0^s(\Omega)$ is denoted by $H^{-s}(\Omega)$. For a fixed positive real number T, and a Banach space X, we denote by $L^p(X)$, $H^s(X)$ and $C(X)$ the space $L^p(0, T; X)$, $H^s(0, T; X)$ and $C(0, T; X)$, respectively.

To formulate the problem (1.2) in a variational form, we shall find the displacement $\boldsymbol{u}(t)$ in $H_0^1(\Omega)^d$ and the divergence field $\phi(t)$ in $L_0^2(\Omega) = \{q \in L^2(\Omega), \int_\Omega q = 0\}$. Furthermore, we set

$$X = H_0^1(\Omega)^d, \qquad V = \{\boldsymbol{v} \in L^2(\Omega)^d : \boldsymbol{v} \cdot n|_\Gamma = 0\}. \tag{2.1}$$

Then for given $\boldsymbol{f} \in W^{2,\infty}(L^2(\Omega)^d)$ and given initial displacement field $\boldsymbol{u}_0 \in X \cap H^2(\Omega)^d$ and initial velocity field $\boldsymbol{u}_t^0 \in X \cap H^2(\Omega)^d$, the variational formulation of problem (1.2) is as follows: seek a pair $(\boldsymbol{u}, \phi)$:

$$\boldsymbol{u} \in L^\infty(V) \cap L^2(X), \qquad \frac{\partial^2 \boldsymbol{u}}{\partial t^2} \in L^2(V^{-1}(\Omega)^d), \qquad \phi \in L^2(L_0^2(\Omega)) \tag{2.2}$$

such that

$$\begin{cases} \left(\dfrac{\partial^2 \boldsymbol{u}}{\partial t^2}, \boldsymbol{v}\right) + (\mu \nabla \boldsymbol{u}, \nabla \boldsymbol{v}) - (\phi, \nabla \cdot \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in H_0^1(\Omega)^d, \\ (\nabla \cdot \boldsymbol{u}, q) = -\eta(\phi, q), \quad \forall q \in L_0^2(\Omega), \end{cases} \tag{2.3}$$

with initial conditions $\boldsymbol{u}(0) = \boldsymbol{u}_0, \boldsymbol{u}_t(0) = \boldsymbol{u}_1$.

It is known that there is some $T > 0$ for which problem (2.3) has at least a solution. In the following, we shall assume that the solution to (2.3) exists for all times and it is as smooth as needed.