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# A note on partial functional differential equations with state-dependent delay

Eduardo Hernández*, Andréa Prokopczyk, Luiz Ladeira

*ICMC, Universidade de São Paulo, São Carlos, Cx 668, 13560-970 SP, Brazil*

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**Abstract**

In this paper we study the existence of mild solutions for a class of abstract partial functional differential equation with state-dependent delay.


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*Keywords:* Functional differential equations; Mild solutions; State-dependent delay; Unbounded delay

## 1. Introduction

In this note we establish the existence of mild solutions for a abstract Cauchy problem described in the form

$$x'(t) = Ax(t) + f(t, x_{\rho(t,x_t)}), \quad t \in I = [0, a], \tag{1}$$

$$x_0 = \varphi \in \mathscr{B}, \tag{2}$$

where $A$ is the infinitesimal generator of a compact $C_0$-semigroup of bounded linear operators $(T(t))_{t \geqslant 0}$ on a Banach space $X$; the function $x_s : (-\infty, 0] \to X$, $x_s(\theta) = x(s + \theta)$, belongs to some abstract phase space $\mathscr{B}$ described axiomatically and $f : I \times \mathscr{B} \to X$, $\rho : I \times \mathscr{B} \to (-\infty, a]$ are appropriate functions.

* Corresponding author. Tel.: +55 16 3373 9754; fax: +55 16 3371 2238.

*E-mail addresses:* lalohm@icmc.sc.usp.br (E. Hernández), andreia@icmc.sc.usp.br (A. Prokopczyk), ladeira@icmc.sc.usp.br (L. Ladeira).

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The literature related to ordinary and partial functional differential equations with delay for which $\rho(t, \psi) = t$ is very extensive and we refer the reader to Hale and Lunel [7] and Wu [17] concerning this matter.

Functional differential equations with state-dependent delay appear frequently in applications as model of equations and for this reason the study of this type of equations has received great attention in the last years, see for instance [1–5,8–11,13,14,16,18] and the references therein. The literature devoted to this subject is concerned with functional differential equations which the state belong to some finite-dimensional space. Obviously, this type of systems does not permit the study of partial functional differential equations with state-dependent delay. This fact is the main motivation of our paper.

Throughout this paper, $A : D(A) \subset X \to X$ is the infinitesimal generator of a compact semigroup of linear operators $(T(t))_{t \geqslant 0}$ on a Banach space $X$ and $\widetilde{M}$ is a constant such that $\|T(t)\| \leqslant \widetilde{M}$ for every $t \in I$. Related semigroup theory, we suggest the Pazy book [15].

In this work we will employ an axiomatic definition for the phase space $\mathscr{B}$ which is similar to those introduced in [12]. Specifically, $\mathscr{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into $X$ endowed with a seminorm $\|\cdot\|_{\mathscr{B}}$ and satisfying the following axioms:

(**A**) If $x : (-\infty, b] \to X$, $b > 0$, is continuous on $[0, b]$ and $x_0 \in \mathscr{B}$, then for every $t \in [0, b]$ the following conditions hold:
   (a) $x_t$ is in $\mathscr{B}$.
   (b) $\|x(t)\| \leqslant H\|x_t\|_{\mathscr{B}}$.
   (c) $\|x_t\|_{\mathscr{B}} \leqslant M(t)\|x_0\|_{\mathscr{B}} + K(t)\sup\{\|x(s)\| : 0 \leqslant s \leqslant t\}$,
   where $H > 0$ is a constant; $K, M : [0, \infty) \to [1, \infty)$, $K(\cdot)$ is continuous, $M(\cdot)$ is locally bounded and $H, K(\cdot), M(\cdot)$ are independent of $x(\cdot)$.

(**A1**) For the function $x(\cdot)$ in (**A**), $x_t$ is a $\mathscr{B}$-valued continuous function on $[0, b]$.

(**B**) The space $\mathscr{B}$ is complete.

**Example** (*The phase space* $C_r \times L^p(g; X)$). Let $g : (-\infty, -r) \to \mathbb{R}$ be a positive Lebesgue integrable function and assume that there exists a non-negative and locally bounded function $\gamma$ on $(-\infty, 0]$ such that $g(\xi + \theta) \leqslant \gamma(\xi) g(\theta)$ for all $\xi \leqslant 0$ and $\theta \in (-\infty, -r) \backslash N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set with Lebesgue measure zero. The space $C_r \times L^p(g; X)$ consists of all classes of functions $\varphi : (-\infty, 0] \to X$ such that $\varphi$ is continuous on $[-r, 0]$, Lebesgue measurable and $g\|\varphi\|^p$ is Lebesgue integrable on $(-\infty, -r)$. The seminorm in $C_r \times L^p(g : X)$ is defined by

$$\|\varphi\|_{\mathscr{B}} := \sup\{\|\varphi(\theta)\| : -r \leqslant \theta \leqslant 0\} + \left(\int_{-\infty}^{-r} g(\theta)\|\varphi(\theta)\|^p \, \mathrm{d}\theta\right)^{1/p}.$$

Assume that $g(\cdot)$ verifies the conditions (g-5), (g-6) and (g-7) in the nomenclature of [12]. In this case, $\mathscr{B} = C_r \times L^p(g; X)$ verifies axioms (**A**), (**A1**), (**B**) see [12, Theorem 1.3.8] for details. Moreover, when $r = 0$ and $p = 2$ we have that $H = 1$, $M(t) = \gamma(-t)^{1/2}$ and $K(t) = 1 + (\int_{-t}^{0} g(\theta)\, \mathrm{d}\theta)^{1/2}$ for $t \geqslant 0$.