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On the structure of the energy conserving low-order models and their relation to Volterra gyrostat

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Abstract

Low-order models (LOM) described by a system of nth-order (nonlinear) ordinary differential equations (ODE) of the type

$$\dot{x}_i = \mathbf{x}^{\mathrm{T}} A^{(i)} \mathbf{x} + B_i \mathbf{x} + c_i, \quad i = 1, 2, \ldots, n$$

(where $\mathbf{x}$ is a column vector, $A^{(i)}$ is a $n \times n$ matrix, B_i is a row vector, c_i is a scalar and T denotes the transpose) routinely arise when we apply the Galerkin type projection techniques to the quasi-geostrophic potential vorticity equation (with forcing, dissipation and topography), Rayleigh–Bernard convection and Burgers' equation, to mention a few. To our knowledge there is no systematic method for testing if a given LOM conserves energy. Our goal in this paper is twofold. First, we derive a set of sufficient conditions on the structural parameters ($A^{(i)}$, B_i and c_i for $i = 1, 2, \ldots, n$) for conserving energy. It is well known in Mathematical Physics that the Volterra gyrostat and many of its special cases including the Euler gyroscope represent a prototype of energy conserving dynamical systems. It turns out that a special case of our sufficient condition is closely related to the Volterra gyrostats. Exploiting this relation, we then derive an algorithm for rewriting the LOM (corresponding to the special case of our sufficient conditions) as a system of coupled gyrostats which brings out the inherent relation between the energy conserving LOM and the system of coupled gyrostats.

Keywords: Low order models; Energy conserving; Sufficient conditions; Structural parameters; Volterra gyrostat

1. Introduction

Let $\mathbf{x} = (x_1, x_2, \ldots, x_{\mathrm{n}})^{\mathrm{T}} \in \Re^n$ be a (real) column vector denoting the state of a dynamical system whose time evolution is governed by a system of nonlinear (quadratic type) coupled ordinary differential equations (ODEs) given by

$$\dot{x}_i = \mathbf{x}^{\mathrm{T}} A^{(i)} \mathbf{x} + B_i \mathbf{x} + c_i, \tag{1.1}$$

where $\dot{x}_i = \mathrm{d}x_i/\mathrm{d}t$, $A^{(i)} \in \Re^{n \times n}$ is a (real) matrix, $B_i \in \Re^{1 \times n}$ is a (real) row vector and $c_i \in \Re$ is a (real) constant. Equivalently, in vector form (1.1) can be written as

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x}, \mathbf{x}) + B\mathbf{x} + \mathbf{c}, \tag{1.2}$$

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where $A(\mathbf{x},\mathbf{x}) = (\mathbf{x}^{\mathrm{T}}A^{(1)}\mathbf{x}, \mathbf{x}^{\mathrm{T}}A^{(2)}\mathbf{x}, \ldots, \mathbf{x}^{\mathrm{T}}A^{(n)}\mathbf{x})^{\mathrm{T}} \in \Re^n$ is a vector of quadratic forms in $\mathbf{x}$, $B \in \Re^{n\times n}$ is a matrix with B_i as its ith row and $\mathbf{c}=(c_1, c_2, \ldots, c_{\mathrm{n}})^{\mathrm{T}} \in \Re^n$. An example of a typical system that arises in rigid body mechanics and in meteorology is given by

$$
\begin{aligned}
\dot{x}_1 &= a_1x_2x_3 - \sum_{j=1}^{3} b_{1j}x_j + c_1,\\
\dot{x}_2 &= a_2x_1x_3 - \sum_{j=1}^{3} b_{2j}x_j + c_2,\\
\dot{x}_3 &= a_3x_1x_2 - \sum_{j=1}^{3} b_{3j}x_j + c_3,
\end{aligned}
\tag{1.3}
$$

where

$$
A^{(1)} = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & a_1\\ 0 & 0 & 0 \end{pmatrix}, \quad A^{(2)} = \begin{pmatrix} 0 & 0 & a_2\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{pmatrix}, \quad A^{(3)} = \begin{pmatrix} 0 & a_3 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{pmatrix}.
$$

Let E: $\Re^n \to \Re$ be given by

$$
\mathrm{E}(\mathbf{x}) = \frac{1}{2}\sum_{i=1}^{n} K_i x_i^2 = \frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{K}\mathbf{x}, \tag{1.4}
$$

where $K_i > 0$ and $\mathbf{K}$ is a diagonal matrix

$$
\mathbf{K} = \mathrm{Diag}(K_1, K_2, \ldots, K_{\mathrm{n}}). \tag{1.5}
$$

As a positive definite quadratic form, $\mathrm{E}(\mathbf{x})$ in (1.4) denotes (generalized) energy.

Our first goal is to derive a set of sufficient conditions on the structural parameters—matrix $A^{(i)}$, row vector B_i and the scalar c_i for $i = 1, 2, \ldots, n$ of system (1.1) such that the time derivative $\dot{\mathrm{E}}(\mathbf{x})$ of $\mathrm{E}(\mathbf{x})$ evaluated along the trajectory of (1.1) vanishes (identically in $\mathbf{x}$). That is, we are seeking conditions under which (1.1) conserves the energy. One of the main results is contained in Theorem 2.1 in Section 2.

A little reflection would reveal that we can assume one of the K_i's is a fixed constant. Without loss of generality, it is assumed that $K_1 = 1$. All the formulas for K_i are conditioned on this assumption. If, instead, we assume that $K_i = 1$ for $i \neq 1$, we would obtain a corresponding equivalent set of formulas. Refer to Example 3.3.

Equation of the type (1.2) are called low-order models (LOMs) and routinely arise from the application of Galerkin type projection techniques [21] to the standard models of interest in fluid mechanics and atmospheric sciences including the Rayleigh–Bérnard convection equations [20,5], quasi-geostrophic potential vorticity equation [2,3,16–18,12] and Burgers' equation [19]. It is well known that while the order n of the resulting LOM depends on the number of modes, the structure of the resulting matrices $A^{(i)}$ $(i = 1, 2, \ldots, n)$, B and the vector $\mathbf{c}$ depends critically on the type of modes ($\sin 2x$ or $\sin 4x$) in the orthogonal expansion of the field variable.

Despite this wide popularity of LOM, it seems that there is no guiding principle for the choice of the number and type of modes leading to LOM that will preserve energy. For example, Howard and Krishnamurthi [9] analyzed a LOM of order six (hereafter call HK [9] model) for the Rayleigh–Bérnard convection. It turned out that this model while useful, did not conserve the energy $\mathrm{E}(\mathbf{x})$ in (1.4). Later Hermiz et al. [8] and Thiffeault and Horton [22], by adding more harmonic terms in the spectral expansion for the stream function and the temperature field, obtained an improved version of the HK [9] model that conserved $\mathrm{E}(\mathbf{x})$. Refer to Gluhovsky et al. [7] for details. Our goal in this paper is to replace this *ad hoc* approach by a systematic methodology by which one can test if a LOM resulting from this exercise