Nonlinear Analysis 145 (2016) 1–23

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Global a priori bounds for weak solutions to quasilinear parabolic equations with nonstandard growth

Patrick Winkert[a,*], Rico Zacher[b]

[a] Technische Universität Berlin, Institut für Mathematik, Straße des 17. Juni 136, 10623 Berlin, Germany
[b] Universität Ulm, Institut für Angewandte Analysis, Helmholtzstraße 18, 89069 Ulm, Germany

ARTICLE INFO

Article history:
Received 11 December 2015
Accepted 29 June 2016
Communicated by S. Carl

MSC:
35K59
35K92

Keywords:
Quasilinear parabolic equations
Nonlinear boundary condition
Nonstandard growth
Variable exponents
De Giorgi iteration
A priori bounds

ABSTRACT

In this paper we study a rather wide class of quasilinear parabolic problems with nonlinear boundary condition and nonstandard growth terms. It includes the important case of equations with a $p(t, x)$-Laplacian. By means of the localization method and De Giorgi's iteration technique we derive global a priori bounds for weak solutions of such problems. Our results seem to be new even in the constant exponent case.

1. Introduction

This paper is concerned with a rather wide class of quasilinear parabolic problems with nonlinear boundary condition. An important feature of the problems under study is that they may contain nonlinear terms with variable growth exponents depending on time and space. To be more precise, let $\Omega \subset \mathbb{R}^N, N > 1$, be a bounded domain with Lipschitz boundary $\Gamma := \partial\Omega$ and let $T > 0, Q_T = (0,T) \times \Omega$ and $\Gamma_T = (0,T) \times \Gamma$. Given $p \in C(\overline{Q}_T)$ satisfying $1 < p^- = \inf_{(t,x)\in\overline{Q}_T} p(t,x)$, the main purpose of the paper consists in proving global a priori bounds for weak solutions of parabolic equations of the form

$$\begin{aligned} u_t - \operatorname{div} \mathcal{A}(t,x,u,\nabla u) &= \mathcal{B}(t,x,u,\nabla u) && \text{in } Q_T, \\ \mathcal{A}(t,x,u,\nabla u)\cdot\nu &= \mathcal{C}(t,x,u) && \text{on } \Gamma_T, \\ u(0,x) &= u_0(x) && \text{in } \Omega. \end{aligned} \tag{1.1}$$

* Corresponding author.
E-mail address: winkert@math.tu-berlin.de (P. Winkert).

http://dx.doi.org/10.1016/j.na.2016.06.012

Here $\nu(x)$ denotes the outer unit normal of Ω at $x \in \Gamma$, $u_0 \in L^2(\Omega)$ and the nonlinearities involved $\mathcal{A}: Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$, $\mathcal{B}: Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $\mathcal{C}: \Gamma_T \times \mathbb{R} \to \mathbb{R}$ are assumed to satisfy appropriate $p(t,x)$-structure conditions which are stated in hypothesis (H), see below. Our setting includes as a special case parabolic equations with a $p(t,x)$-Laplacian, which is given by

$$\Delta_{p(t,x)} u = \operatorname{div}\left(|\nabla u|^{p(t,x)-2}\nabla u\right),$$

and which reduces to the $p(x)$-Laplacian if $p(t,x)=p(x)$, respectively, to the well-known p-Laplacian in case $p(t,x)\equiv p$.

Nonlinear equations of the type considered in (1.1) with variable exponents in the structure conditions are usually termed equations with nonstandard growth. Such equations are of great interest and occur in the mathematical modeling of certain physical phenomena, for example in fluid dynamics (flows of electro-rheological fluids or fluids with temperature-dependent viscosity), in nonlinear viscoelasticity, in image processing and in processes of filtration through porous media, see for example, Acerbi–Mingione–Seregin [1], Antontsev–Díaz–Shmarev [7], Antontsev–Rodrigues [8], Chen–Levine–Rao [21], Diening [23], Rajagopal–Růžička [37], Růžička [39] and Zhikov [51,52] and the references therein.

Throughout the paper we impose the following conditions.

(H) The functions $\mathcal{A}: Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$, $\mathcal{B}: Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $\mathcal{C}: \Gamma_T \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions satisfying the subsequent structure conditions:

(H1) $|\mathcal{A}(t,x,s,\xi)| \le a_0|\xi|^{p(t,x)-1} + a_1|s|^{q_1(t,x)\frac{p(t,x)-1}{p(t,x)}} + a_2$, a.e. in Q_T,

(H2) $\mathcal{A}(t,x,s,\xi)\cdot\xi \ge a_3|\xi|^{p(t,x)} - a_4|s|^{q_1(t,x)} - a_5$, a.e. in Q_T,

(H3) $|\mathcal{B}(t,x,s,\xi)| \le b_0|\xi|^{p(t,x)\frac{q_1(t,x)-1}{q_1(t,x)}} + b_1|s|^{q_1(t,x)-1} + b_2$, a.e. in Q_T,

(H4) $|\mathcal{C}(t,x,s)| \le c_0|s|^{q_2(t,x)-1} + c_1$, a.e. in Γ_T,

for all $s \in \mathbb{R}$, all $\xi \in \mathbb{R}^N$ and with positive constants a_i, b_j, c_l. Further, $p \in C(\overline{Q}_T)$ with $\inf_{(t,x)\in\overline{Q}_T} p(t,x) > 1$ and $q_1 \in C(\overline{Q}_T)$ as well as $q_2 \in C(\overline{\Gamma}_T)$ are chosen such that

$$\begin{aligned} p(t,x) &\le q_1(t,x) < p^*(t,x), \quad (t,x)\in\overline{Q}_T,\\ p(t,x) &\le q_2(t,x) < p_*(t,x), \quad (t,x)\in\overline{\Gamma}_T, \end{aligned}$$

with the critical exponents

$$p^*(t,x) = p(t,x)\frac{N+2}{N}, \qquad p_*(t,x) = p(t,x)\frac{N+2}{N} - \frac{2}{N}.$$

(P) The exponent $p \in C(\overline{Q}_T)$ is log-Hölder continuous on Q_T, that is, there exists $k>0$ such that

$$|p(t,x)-p(t',x')| \le \frac{k}{\log\left(e + \frac{1}{|t-t'|+|x-x'|}\right)},$$

for all $(t,x),(t',x') \in Q_T$.

A function $u: Q_T \to \mathbb{R}$ is called a **weak solution** (**subsolution**, **supersolution**) of problem (1.1) if

$$u \in \mathcal{W} := \left\{ v \in C\left([0,T];L^2(\Omega)\right) : |\nabla v| \in L^{p(\cdot,\cdot)}(Q_T) \right\}$$