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On the lack of bound states for certain NLS equations on metric graphs

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ABSTRACT

The purpose of this paper is to prove some results on the absence of bound states for certain nonlinear Schrödinger equations on noncompact metric graphs with localized nonlinearity. In particular, we show how the topological and metric properties of graphs affect the existence/nonexistence of bound states. This work completes the discussion initiated in Serra and Tentarelli (2016) and Tentarelli (2016).

1. Introduction

The study of NLS equations on noncompact metric graphs has gained popularity in the last few years, because (among other things) these equations are expected to describe the dynamics of *Bose–Einstein condensates* in ramified traps (see [13,15]). In particular, many studies concentrate on a specific NLSE, the cubic focusing *Gross–Pitaevskii* equation,

$$i\psi_t = -\psi_{xx} - |\psi|^2\psi \tag{1}$$

on a graph $\mathcal{G}$, with homogeneous *Kirchhoff conditions* at the vertices (see (7)). A central role in this line of research is played by *stationary solutions* of prescribed mass (i.e., L^2 norm) $\mu > 0$, namely functions of the

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form

$$\psi(t,x) = e^{i\lambda t} u(x), \quad u : \mathcal{G} \to \mathbb{C},\ \lambda \in \mathbb{R}, \tag{2}$$

which solve (1) when the function u solves the stationary NLS equation

$$u'' + |u|^2 u = \lambda u$$

(see e.g. [1,5–7]). The functions u with these properties are called *bound states* of mass μ.

The papers [14,17] present several interesting motivations for investigating a variant of this problem, characterized by the fact that the nonlinearity affects only a *compact* part of the graph. One speaks therefore of problems with *localized* nonlinearity. In this case, and for a generic power nonlinearity, the bound states u appearing in (2) satisfy the same mass constraint

$$\int_{\mathcal{G}} |u|^2\, dx = \mu \tag{3}$$

but solve (for some $\lambda \in \mathbb{R}$) the stationary NLS equation

$$u'' + \kappa(x)|u|^{p-2} u = \lambda u \tag{4}$$

on each edge of $\mathcal{G}$, still with Kirchhoff boundary conditions. The coefficient κ is the characteristic function of the part of $\mathcal{G}$ where the nonlinearity is located. Bound states satisfy therefore a *double regime*: nonlinear in a compact part of $\mathcal{G}$ and linear elsewhere. The exponent p is always assumed to be greater than 2; when $p \in (2,6)$, the problem is called L^2-subcritical (see [11]).

In this work we confine ourselves to Kirchhoff boundary conditions. Many other choices (both in the localized and in the non-localized case) are possible, such as, for instance, the case of δ-like conditions at the vertices. Recent results on this topic are presented in [2–4].

In this paper we are mainly concerned with problems with localized nonlinearity. Specifically, we consider a noncompact metric graph $\mathcal{G}$ and we assume that the nonlinearity is localized in the compact core $\mathcal{K}$ of $\mathcal{G}$, namely the subgraph of $\mathcal{G}$ consisting of its bounded edges (see Section 2 for precise statements).

Thus a bound state of mass μ for the NLS equation on $\mathcal{G}$ with nonlinearity localized on $\mathcal{K}$ is a function u that satisfies the mass constraint (3) and solves Eq. (4) on each edge of $\mathcal{G}$, with Kirchhoff boundary conditions at each vertex of $\mathcal{K}$.

Defining $H^1_\mu(\mathcal{G}) = \{u \in H^1(\mathcal{G}) :\ \|u\|^2_{L^2(\mathcal{G})} = \mu\}$, it is immediate to recognize (see Section 2) that bound states correspond to critical points on $H^1_\mu(\mathcal{G})$ of the energy functional

$$E(u) = \frac{1}{2}\int_{\mathcal{G}} |u'|^2\, dx - \frac{1}{p}\int_{\mathcal{K}} |u|^p\, dx.$$

If u happens to be not only a critical point of the functional E but an absolute minimizer, it is called a *ground* state.

Existence (and multiplicity) of ground and bound states for the NLS equation with localized nonlinearity has been studied in the papers [19,20], in dependence of the parameters μ and p. We summarize in the next theorem the main results obtained so far in order to explain our motivations.

Theorem 1.1 (*[19,20]*). *Let $\mathcal{G}$ be a noncompact metric graph with nonempty compact core.*

1. *If $p \in (2,4)$, for every $\mu > 0$ there exists a ground state of mass μ.*
2. *If $p \in (2,6)$, for every μ large there exist many bound states of mass μ.*
3. *If $p \in [4,6)$, for every μ large there exists a ground state of mass μ.*
4. *If $p \in [4,6)$, for every μ small there exist no ground states of mass μ.*