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Free-surface capillary–gravity azimuthal equatorial flows

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ABSTRACT

We dedicate this paper to the investigation of certain special types of geophysical water flows arising in the equatorial ocean which are driven by gravity and surface tension. To be more specific, we consider steady flows – written in cylindrical coordinates – which are moving in the azimuthal direction, with no variation in this direction, and which exhibit a vertical structure and allow for a surface distortion. Using a functional analytic approach we prove that any small deviation of the pressure function from the pressure required to maintain the free surface flat and undisturbed gives rise to a genuine wave solution.

1. Introduction

We are motivated in our study by the intriguing features the ocean exhibits in the Equatorial region, situated within a band of about 2° latitude from the Equator. One of the many peculiarities is the strong mean ocean vertical stratification in the tropical zone of the ocean, greater than anywhere else in the ocean (as evidentiated by [10]) and giving rise to a sharp near-surface interface (called thermocline) which separates a shallow layer of relatively warm, less denser water near the surface, from a deeper layer of colder and therefore denser water. Another special feature of the equatorial region is the presence of the underlying currents, cf. [24,26], which in a strip of approximately 300 km wide about the Equator present a pronounced depth-dependence of the following structure: while in a layer adjacent to the surface, of about 100 m height, there is westward drift that is triggered by the prevailing trade winds, right below this layer lies the Equatorial Undercurrent (EUC), an eastward pointing stream, residing on the thermocline and which

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plays a major role in the geophysical ocean dynamics in the equatorial region [8,10,25] and which is credited with the formation of the El Niño and La Niña phenomena.

The lessening of the Coriolis forces along the Equator and the preferred direction of propagation (East–West) of three dimensional ocean waves in the equatorial regime allow the employment of more tractable (nevertheless still nonlinear) approximations to the full governing equations, the so-called β-plane and f-plane approximations, respectively.

A starting point in gaining insight into the dynamics of the Equatorial region from a rigorous mathematical perspective was the derivation by Constantin [4] of an exact and explicit solution (in the Lagrangian framework) to the β-plane governing equations. Following [4] there was a considerable amount of works concerning the study of water flows in the Equatorial region and treating equatorially trapped waves (so called Kelvin waves) [3,11], internal waves of large amplitude [5,7,12], instability of surface or internal equatorial waves [6,14,15,18] or – concerning the preferred propagation direction mentioned above – azimuthal waves in cylindrical [9] and spherical coordinates [21].

The impressive effort invested in deriving exact solutions, cf. [4,13,17,16,19,22,23], is rooted in the need for more directly relevant analyses. To be more specific, useful approximate systems were obtained from exact solutions by perturbation procedures, cf. [20].

Our aim here is to derive an exact solution, in a cylindrical frame, describing a flow propagating in the azimuthal direction acted upon by gravity and surface tension. Our solution also incorporates a suitable distortion of the free surface and exhibits the vertical stratification very much present in the tropical zone of the ocean.

2. The governing equations

We begin this section by a description of the geometry and of the variables defining the associated rotating system. In our choice of a coordinate system the Equator is "straightened" and replaced by a line parallel to the z-axis, while the body of the sphere is represented by a circular disc described in the corresponding polar coordinates. Thus, in a right handed system, our coordinates are (r, θ, z), where r is the distance to the center of the disc (representing the Earth), $\theta \in (-\pi/2, \pi/2)$ is increasing from North to South and measures the deflection from the Equator, and the positive z-axis points from West to East. The line of the Equator is chosen to be associated with $\theta = 0$. The corresponding unit vectors in the (r, θ, z) system are $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$ and the velocity components are (u, v, w).

The governing equations in a coordinate system with its origin at the center of the sphere are Euler's equations, which in cylindrical coordinates are written, cf. [9], as

$$\begin{aligned} u_t + uu_r + \frac{v}{r}u_\theta + wu_z - \frac{v^2}{r} &= -\frac{1}{\rho}p_r + F_r \\ v_t + uv_r + \frac{v}{r}v_\theta + wv_z + \frac{uv}{r} &= -\frac{1}{\rho}\frac{1}{r}p_\theta + F_\theta \\ w_t + uw_r + \frac{v}{r}w_\theta + ww_z &= -\frac{1}{\rho}p_z + F_z, \end{aligned} \tag{2.1}$$

(where $p(r, \theta, z)$ denotes the pressure in the fluid and (F_r, F_θ, F_z) is the body-force vector) and the equation of mass conservation

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}v_\theta + w_z = 0. \tag{2.2}$$

To include the effects of the Earth's rotation in our setting we associate $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$ to a point fixed on the sphere which is rotating about its polar axis. This means that we need to add in the left of Euler's equations