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Existence and orbital stability of standing waves for nonlinear Schrödinger systems

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ABSTRACT

In this paper we investigate the existence of solutions in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ for nonlinear Schrödinger systems of the form

$$\begin{cases} -\Delta u_1 = \lambda_1 u_1 + \mu_1 |u_1|^{p_1-2} u_1 + r_1 \beta |u_1|^{r_1-2} u_1 |u_2|^{r_2}, \\ -\Delta u_2 = \lambda_2 u_2 + \mu_2 |u_2|^{p_2-2} u_2 + r_2 \beta |u_1|^{r_1} |u_2|^{r_2-2} u_2, \end{cases}$$

under the constraints

$$\int_{\mathbb{R}^N} |u_1|^2\, dx = a_1 > 0, \quad \int_{\mathbb{R}^N} |u_2|^2\, dx = a_2 > 0.$$

Here $N \geq 1, \beta > 0, \mu_i > 0, r_i > 1, 2 < p_i < 2 + \frac{4}{N}$ for $i = 1, 2$ and $r_1 + r_2 < 2 + \frac{4}{N}$. This problem is motivated by the search of standing waves for an evolution problem appearing in several physical models. Our solutions are obtained as constrained global minimizers of an associated functional. Note that in the system λ_1 and λ_2 are unknown and will correspond to the Lagrange multipliers. Our main result is the precompactness of the minimizing sequences, up to translation. Assuming the local well posedness of the associated evolution problem we then obtain the orbital stability of the standing waves associated to the set of minimizers.

1. Introduction

We consider the existence of solutions to a nonlinear Schrödinger system of the form

$$\begin{cases} -\Delta u_1 = \lambda_1 u_1 + \mu_1 |u_1|^{p_1-2} u_1 + r_1 \beta |u_1|^{r_1-2} u_1 |u_2|^{r_2}, \\ -\Delta u_2 = \lambda_2 u_2 + \mu_2 |u_2|^{p_2-2} u_2 + r_2 \beta |u_1|^{r_1} |u_2|^{r_2-2} u_2, \end{cases} \tag{1.1}$$

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satisfying the conditions

$$\int_{\mathbb{R}^N} |u_1|^2\, dx = a_1, \qquad \int_{\mathbb{R}^N} |u_2|^2\, dx = a_2. \tag{1.2}$$

Here $a_1, a_2 > 0$ are prescribed and we shall assume throughout the paper

(H0) $N \geq 1, \beta > 0, \mu_i > 0, r_i > 1, 2 < p_i < 2 + \frac{4}{N}$ for $i = 1, 2$ and $r_1 + r_2 < 2 + \frac{4}{N}$.

The problem under consideration is associated to the research of standing waves, namely, solutions having the form

$$\Psi_1(t,x) = e^{-i\lambda_1 t} u_1(x), \qquad \Psi_2(t,x) = e^{-i\lambda_2 t} u_2(x),$$

for some $\lambda_1, \lambda_2 \in \mathbb{R}$, of the nonlinear Schrödinger system

$$\begin{cases} -i\partial_t \Psi_1 = \Delta \Psi_1 + \mu_1 |\Psi_1|^{p_1 - 2} \Psi_1 + \beta |\Psi_1|^{r_1 - 2} \Psi_1 |\Psi_2|^{r_2}, \\ -i\partial_t \Psi_2 = \Delta \Psi_2 + \mu_2 |\Psi_2|^{p_2 - 2} \Psi_2 + \beta |\Psi_1|^{r_1} |\Psi_2|^{r_2 - 2} \Psi_2, \end{cases} \quad \text{in } \mathbb{R} \times \mathbb{R}^N. \tag{1.3}$$

This system comes from mean field models for binary mixtures of Bose–Einstein condensates or for binary gases of fermion atoms in degenerate quantum states (Bose–Fermi mixtures, Fermi–Fermi mixtures), see [2,12,22].

One motivation to look for normalized solutions of system (1.1) is that the masses

$$\int_{\mathbb{R}^N} |\Psi_1|^2\, dx \quad \text{and} \quad \int_{\mathbb{R}^N} |\Psi_2|^2\, dx$$

are preserved along the trajectories of (1.3). Our solutions of (1.1)–(1.2) will be obtained as minimizers of the functional

$$J(u_1, u_2) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_1|^2 + |\nabla u_2|^2\, dx - \int_{\mathbb{R}^N} \frac{\mu_1}{p_1} |u_1|^{p_1} + \frac{\mu_2}{p_2} |u_2|^{p_2} + \beta |u_1|^{r_1} |u_2|^{r_2}\, dx$$

constrained on

$$S(a_1, a_2) := \{(u_1, u_2) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) : \|u_1\|_2^2 = a_1, \|u_2\|_2^2 = a_2\}.$$

Namely we are to consider the minimization problem

$$m(a_1, a_2) := \inf_{(u_1, u_2) \in S(a_1, a_2)} J(u_1, u_2). \tag{1.4}$$

It is standard that the minimizers of (1.4) are solutions to (1.1)–(1.2) where λ_1, λ_2 appear as the Lagrange multipliers. Actually the existence of minimizers for (1.4) will be obtained as a consequence of the stronger statement that any minimizing sequence for (1.4) is, up to translation, precompact.

Theorem 1.1. *Assume* (H0). *Then for any $a_1 > 0$ and $a_2 > 0$ all minimizing sequences for* (1.4) *are precompact in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ after a suitable translation.*

Following some initial works [30,31], the compactness concentration principle of P.L. Lions [19,20] has had, over the last thirty years, a deep influence on solving minimization problems under constraints. Heuristic arguments readily convince that in our problem the compactness of any minimizing sequence holds if the following strict subadditivity conditions are satisfied.

$$m(a_1, a_2) < m(b_1, b_2) + m(a_1 - b_1, a_2 - b_2), \tag{1.5}$$

where $0 \leq b_i < a_i$ for $i = 1, 2$ and $b_1 + b_2 \neq 0$.