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Blow-up results to certain hyperbolic model problems in fluid mechanics

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ABSTRACT

Global existence theorems for large data to hyperbolic fluid models very often have restrictions on the size of the relaxation parameter depending on the size of the initial data, and therefore leave the general question of well-posedness for large data open. In this note we show blow-up results to some corresponding model problems that underline the complexity of this question and hence also show the strength of global existence theorems for small parameters.

1. Introduction

The global strong well-posedness of the classical Navier–Stokes equation

$$\begin{aligned} v_t + (v \cdot \nabla)v - \mu\Delta v + \nabla p &= 0 \qquad \text{in } (0,T) \times \mathbb{R}^2, \\ \operatorname{div} v &= 0 \qquad \text{in } (0,T) \times \mathbb{R}^2, \\ v(0,\cdot) &= v^0 \qquad \text{in } \mathbb{R}^2 \end{aligned} \tag{NS}$$

is well-known, whereas the situation for hyperbolic models that arise by introducing certain relaxations is not yet settled. For global existence theorems for large data of these models it seems to be quite common to have restrictions on the size of the relaxation parameter depending on the size of the initial data. Roughly speaking, corresponding theorems state that if the product of the parameter times some norm of the initial data is small, then there is a global solution. On the one hand, in applications one usually thinks of small relaxations, but on the other hand, the general question of well-posedness for large data remains open. In this note we want to study some model problems that have blow-ups in finite time if the product of the parameter times some norm of the initial data is big.

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The following are some examples of global existence theorems with a constraint on the relaxation parameter. Let $T, \tau, \mu > 0$. For the model

$$\begin{aligned} \tau u_{tt} + u_t + (u\cdot\nabla)u - \mu\Delta u + \nabla p &= 0 \qquad \text{in } (0,T)\times\mathbb{R}^2, \\ \operatorname{div} u &= 0 \qquad \text{in } (0,T)\times\mathbb{R}^2, \\ u(0,\cdot) = u^0, \qquad u_t(0,\cdot) &= u^1 \qquad \text{in } \mathbb{R}^2 \end{aligned} \tag{sHNS}$$

there are quite a few results, namely [1, Theorem 2.2], [10, Theorem 0.1] and [6, Theorem 1], but all of them have the described constraint on the relaxation parameter τ. Another example is given in [5], where for $\alpha > 0$ the equation

$$\begin{aligned} &\tau u_{tt} + u_t + (u\cdot\nabla)u - \mu\Delta u - \tfrac{1}{\alpha}\nabla \operatorname{div} u = 0 \quad \text{in } (0,T)\times\mathbb{R}^2, \\ &u(0,\cdot) = u^0, \qquad u_t(0,\cdot) = u^1 \qquad\qquad\qquad \text{in } \mathbb{R}^2 \end{aligned} \tag{1}$$

is studied. The global existence theorem [5, Theorem 2] imposes analogue conditions on τ and α.

The interesting point about this model is the fact, that for $\alpha \to 0$ it is an approximation of (sHNS) and the solution has finite propagation speed. Therefore we will be able to apply the theory of [17] to proof a blow-up result in Section 2.1.

A further example for a global existence theorem for large data and small relaxation parameter is given in ([15], [16, Section 2.2]), where the so called hyperbolic Navier–Stokes equation

$$\begin{aligned} \tau u_{tt} - \mu\Delta u + u_t + \nabla p + \tau\nabla p_t &= -(u\cdot\nabla)u - (\tau u_t\cdot\nabla)u - (\tau u\cdot\nabla)u_t \qquad \text{in } (0,T)\times\mathbb{R}^2, \\ \operatorname{div} u &= 0 \qquad \text{in } (0,T)\times\mathbb{R}^2, \\ u(0,\cdot) = u^0, \qquad u_t(0,\cdot) &= u^1 \qquad \text{in } \mathbb{R}^2 \end{aligned} \tag{HNS}$$

is studied (also cp. [13,14]). The main difference to the models above and the classical Navier–Stokes equation is of course the quasilinearity of the problem, which makes it more complicated concerning well-posedness.

To classify these global existence results, a first thing to remember about relaxed systems is the work [2], where it was shown that delayed systems that are formally close to the original one, can behave differently. More precise, in [2, Theorem 1.3] it was shown that formal high Taylor expansions of the delayed term, can lead to ill-posedness and therefore it is not clear what to expect in general.

In this note we prove some blow-up results to model problems corresponding to (sHNS) resp. to (HNS) that have two main consequences for the observations above. First of all they show the strength of global existence theorem for small parameters, since obviously one cannot expect to have a global solution for large data and large parameters in general. Secondly, the results underline the complexity of the question of well-posedness in the spirit of the works [9,3,18]. There the authors constructed blow-up models similar to the Navier–Stokes equation to show that one cannot expect to solve the millennium problem by a general method, since otherwise this would be applicable to the blow-up models which would yield a contradiction. In the same way the results of this note show that a proof of a global existence theorem for large data and large parameters to the problems above must take the specific properties of the equation into account.

The paper is divided into two parts. The Section 1 deals with the well-posedness and the blow-up result of the semilinear model problem, whereas the Section 2 comments on the quasilinear problem. These results are part of the author's Ph.D. thesis [16].

This note uses common notation, for instance for a general Banach space X and $\Omega \subseteq \mathbb{R}^n$ a set, $C^m(\Omega, X)$ denotes the space of m-times continuously differentiable functions with values in X. Analogously $L^p(\Omega, X)$ with $1 \le p \le \infty$ denotes the standard Lebesgue space of X-valued functions with norm $\|\cdot\|_p$. For the Hilbert space $L^2(\Omega, X)$ we write $\langle\cdot,\cdot\rangle$ for the scalar product.