Nonlinear Analysis 144 (2016) 77–92

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Existence and multiplicity results for generalized logistic equations

Nguyen Bich Huy[a,*], Bui The Quan[b], Nguyen Huu Khanh[c]

[a] *Applied Analysis Research Group, Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Viet Nam*
[b] *Department of Mathematics, Dong Nai University, 4 Le Quy Don, Bien Hoa, Dong Nai, Viet Nam*
[c] *Department of Mathematics, College of Nature Science, Can Tho University, Viet Nam*

ARTICLE INFO

Article history:
Received 29 December 2015
Accepted 3 June 2016
Communicated by S. Carl

MSC:
35J60
47H07
47H10

Keywords:
Logistic equation
Dependence on the gradient
Fixed point index
Leggett–Williams theorem

ABSTRACT

We consider elliptic Dirichlet problem $-\Delta_p u = \lambda f(x,u,\nabla u) - g(x,u)$ in $\Omega, u = 0$ on $\partial\Omega$. Assume that the nonlinearity f satisfies certain growth condition. Using the fixed point index, the Leggett–Williams theorem on fixed point and the arguments on monotone minorant, we prove the existence and multiplicity results for the equation. This extends some known results.

1. Introduction

Consider the following elliptic Dirichlet problem

$$\begin{cases} -\Delta_p u = \lambda f(x,u,\nabla u) - g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where u is a non-trivial non-negative unknown function, $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain with a smooth boundary $\partial\Omega$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian with $1 < p < N$, and $\lambda > 0$ is a real parameter.

In this paper, we consider Eq. (1.1) under the condition

(f) $|f(x,u,v)| \leqslant m(x)|u|^\alpha + c|v|^\gamma$

* Corresponding author.
E-mail addresses: nguyenbichhuy@tdt.edu.vn (N.B. Huy), buithequan2002@yahoo.com (B.T. Quan).

http://dx.doi.org/10.1016/j.na.2016.06.001
0362-546X/

where $\alpha < p^* - 1, \gamma < \frac{p}{(p^*)'}$ with $p^* = \frac{pN}{N-p}$ and $m \in L^q(\Omega)$ with $q > (\frac{p^*}{1+\alpha})'$. Here, for $t \in (1, \infty)$ the number t' is defined by $\frac{1}{t} + \frac{1}{t'} = 1$. Depending on the value of the exponent α, Eq. (1.1) under the condition (f) is divided into three types: the sub-, equi- or super-diffusive cases corresponding to $\alpha < p-1, \alpha = p-1$ or $\alpha > p-1$, respectively.

Such equations include p-logistic equations (or logistic equation when $p = 2$) corresponding to the case $f(x, u, v) = a(x)u^\alpha, g(x, u) = b(x)u^\beta$ with $1 < \alpha < \beta$. Note that the theory of p-logistic equation has been received a great deal of attention and has had a wide range of applications from reaction–diffusion processes to population dynamics. For further discussions, we refer the reader to [3,5–8,13,11,14,12,16–18,20,21,25,24,26] and references therein. Further, the analogous results have proved in [13,11,14,12] for the case, when p-Laplacian operator was replaced by a non-homogeneous differential operator div $a(\nabla u)$.

Recently, the equation $-\Delta_p u = \lambda f(x, u) - g(x, u)$ has been studied in [3,18] for the super-diffusive case. Using the variational method, the authors obtained the existence of a number $\lambda_* > 0$ such that the problem has at least two positive solutions for $\lambda > \lambda_*$ at least one positive solution for $\lambda = \lambda_*$, and has no positive solution for $\lambda < \lambda_*$.

In this paper, using the fixed point index technique and the cone theoretic arguments, we obtain the following existence and multiplicity results for Eq. (1.1) which are main results of this paper.

(a) the equation has at least one positive solution for all $\lambda > 0$ in the sub-diffusive case and for $\lambda > \lambda_0$ in the equi-diffusive case, where λ_0 is the principal eigenvalue of an associated weighted eigenvalue problem.
(b) the equation has at least two positive solutions in the super-diffusive case for sufficiently large λ.

Note that in [3,18], the authors obtained the existence results for the super-diffusive case with the non-linearity f depending on x and u only. In this sense, our paper generalizes to consider the more general class of non-linearity f and fill the gap by giving the existence results not only for the super-diffusive case but also for the sub-diffusive and equi-diffusive cases.

Eq. (1.1) with the non-linearity f containing ∇ was investigated in [19,23]. However, the condition on f in our paper is slightly weaker. Moreover, both the works [19,23] proved the existence for a positive solution only, but the multiplicity existence result. In comparison with [19,23], our results on multiplicity existence results are new.

We would like to describe briefly the technical elements used in the paper. In the sub- and equi-diffusive cases we consider not necessarily bounded solutions and therefore, there appeared some difficulties in their estimates. We use the arguments of compactness in conjunction with a kind of monotone minorant arguments to overcome the difficulties. To study the super-diffusive case, we apply a variant of the Leggett–Williams theorem on fixed point in cone. This approach is new in the literature.

The paper is organized as follows. In Section 2, we review some preliminary results on fixed point index and reduce Eq. (1.1) to a fixed point problem. Section 3, is devoted to prove our main results on the existence and multiplicity for Eq. (1.1).

2. Preliminary results

2.1. Equations in ordered spaces

Let E be a Banach space ordered by the cone $K \subset E$, that is, K is a closed convex subset such that $\lambda K \subset K$ for all $\lambda \geq 0$, $K \cap (-K) = \{\theta\}$ and ordering in E is defined by $x \leq y$ iff $y - x \in K$.

If D is a bounded relatively open subset of K and $F : \overline{D} \to K$ is a compact operator such that $F(u) \neq u, \forall u \in \partial D$, then the fixed point index $i(F, D, K)$ of F on D with respect to K is well-defined.