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On the blowup criteria and global regularity for the non-diffusive Boussinesq equations with temperature-dependent viscosity coefficient

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ABSTRACT

In this paper, we consider the Boussinesq equations with temperature-dependent viscosity and zero thermal diffusivity. A blowup criterion of classical solution, depending only on $\|\nabla\mu(\theta)\|_{L^q(0,T;L^p)}$ with $p > 2$ and $1/p + 1/q \leq 1/2$, is obtained for the two-dimensional equations. This is in particular consistent with the results in Chae (2006), Hou and Li (2005), Lai et al. (2011), where the global regularity of the 2D non-diffusive Boussinesq equations with constant viscosity coefficient (i.e., $\mu(\theta) =$ Const.) was proved. A Serrin's type blowup criterion, depending on $\|u\|_{L^s(0,T;L^r)} + \|\nabla\mu(\theta)\|_{L^q(0,T;L^p)}$ with $r, p > 3$, $3/r + 2/s \leq 1$ and $3/p + 2/q \leq 1$, is obtained for the three-dimensional case. We also study the global regularity with large data for the two-dimensional equations in a non-divergence form.

1. Introduction

This paper is mainly concerned with the non-diffusive Boussinesq equations with temperature-dependent viscosity coefficient in the form:

$$\begin{cases} u_t + u \cdot \nabla u + \nabla \Pi = \mathrm{div}(\mu(\theta)\nabla u) + \theta e_n, \\ \theta_t + u \cdot \nabla \theta = 0, \\ \mathrm{div} u = 0, \end{cases} \tag{1.1}$$

with the initial conditions:

$$u(x,0) = u_0(x), \qquad \theta(x,0) = \theta_0(x), \tag{1.2}$$

where $x \in \mathbb{R}^n$ $(n = 2, 3)$ and $t \geq 0$. The unknown functions $u \in \mathbb{R}^n$, θ, and Π denote the velocity, temperature and pressure of the fluid, respectively. As usual, $e_n \triangleq (0, \dots, 0, 1)$, $\nabla \triangleq (\partial_1, \dots, \partial_n)$, and $\mathrm{div} f \triangleq \nabla \cdot f$ for $f = (f^1, \dots, f^n)$, where $\partial_i \triangleq \partial_{x_i}$ with $i = 1, \dots, n$.

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Boussinesq system plays an important role in atmospheric and oceanographic sciences (see [32,35,37]), and has received significant attention in the mathematical fluid dynamics community because of its close connection to the multi-dimensional incompressible flows. Indeed, if $\mu(\theta) = 0$ and $n = 2$, then (1.1) reduces to the inviscid Boussinesq equations which can be identified with three-dimensional axisymmetric Euler equations.

There have been many works devoted to the study of the Boussinesq equations, see, for example, [1–3,5,6, 10,7,9,8,11–26,29,33–36,40–44] and the references therein for both the analytical and the numerical studies. In particular, the global regularity of the two-dimensional Boussinesq equations with large data has attracted considerable attention in the last few years in the case when the viscosity and thermal diffusivity coefficients are fixed non-negative constants. For example, the global regularity for the initial and initial–boundary value problem of the two-dimensional Boussinesq with "partial viscosity" (i.e., viscous but non-diffusive case or diffusive but inviscid case) was studied in [6,20,21,24,27,44]. The anisotropic Boussinesq equations with only vertical or horizontal dissipation and thermal diffusion were considered [2,3,5,34,43]. We also refer to [14,28,22,23,33] for other interesting recent results about the two-dimensional anisotropic equations with only horizontal viscosity or diffusivity and the two-dimensional equations with fractional diffusion, respectively.

The works mentioned above are mainly concerned with the case of fixed viscosity and thermal diffusivity coefficients. However, when the viscosity and thermal diffusivity coefficients depend on the temperature, the mathematical theory of such Boussinesq equations, which could be of great importance due to the large temperature contrast in certain applications, becomes much more complicated and are much less studied. Lorca and Boldrini proved the global existence of weak solutions and the local existence of strong solutions with general data in [30] and the global existence of strong solutions with small data in [31] for the Boussinesq model with temperature-dependent viscosity and thermal diffusivity in $\mathbb{R}^n$:

$$\begin{cases} u_t + u\cdot\nabla u + \nabla\Pi = \operatorname{div}(\mu(\theta)\nabla u) + \theta e_n, \\ \theta_t + u\cdot\nabla\theta = \operatorname{div}(\kappa(\theta)\nabla\theta), \\ \nabla\cdot u = 0. \end{cases} \tag{1.3}$$

Recently, Wang–Zhang [42] and Sun–Zhang [40] obtained the global regularity of the solutions for the initial and initial–boundary value problem of (1.3) in two dimensions, respectively. Li, Pan and Zhang [29] also considered the two-dimensional equations and proved the global regularity for the inviscid-diffusive case (i.e., $\mu(\theta) = 0, \kappa(\theta) > 0$).

The purpose of this paper is to study the global regularity of the non-diffusive Boussinesq equations (1.1) with temperature-dependent viscosity, which is much more complicated than the ones considered in [29,40,42]. To do this, we assume throughout this paper that

$$\mu(\xi) \in C^3(\mathbb{R}) \quad \text{and} \quad \mu(\xi) \geq \underline{\mu} \quad \text{for any } \xi \in \mathbb{R}. \tag{1.4}$$

Before stating the main results, we first introduce the local well-posedness of classical solutions of (1.1), (1.2), which can be essentially obtained by using the fixed point theorem.

Theorem 1.1. *Assume that $(u_0, \theta_0) \in H^3(\mathbb{R}^n)$ with $n = 2, 3$ satisfy $\operatorname{div} u_0 = 0$ and that (1.4) holds. Then there exists a positive time $T_0 > 0$, depending only on $\|u_0\|_{H^3}, \|\theta_0\|_{H^3}$ and $\mu(\cdot)$, such that the initial value problem (1.1), (1.2) has a unique smooth solution (u, Π, θ) satisfying*

$$\begin{cases} u \in L^\infty(0,T_0;H^3(\mathbb{R}^n)) \cap L^2(0,T_0;H^4(\mathbb{R}^n)), \quad u_t \in L^\infty(0,T;H^1(\mathbb{R}^n)), \\ \theta \in L^\infty(0,T_0;H^3(\mathbb{R}^n)), \quad \theta_t \in L^\infty(0,T_0;H^2(\mathbb{R}^n)), \quad \nabla\Pi \in L^\infty(0,T_0;H^1(\mathbb{R}^n)). \end{cases} \tag{1.5}$$

It is an outstanding challenging open problem that whether the unique local solution obtained in Theorem 1.1 can exist globally or not. If the answer is negative, then it simultaneously raises the interesting