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Weakly coupled mean-field game systems

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ABSTRACT

Here, we prove the existence of solutions to first-order mean-field games (MFGs) arising in optimal switching. First, we use the penalization method to construct approximate solutions. Then, we prove uniform estimates for the penalized problem. Finally, by a limiting procedure, we obtain solutions to the MFG problem.

1. Introduction

The mean-field game (MFG) framework [34–37,39,38] is a class of methods used to study large populations of rational, non-cooperative agents. MFGs have been the focus of intense research, see, for example, the surveys [28,31]. Here, we investigate MFGs that arise in optimal switching. These games are given by a weakly coupled system of Hamilton–Jacobi equations of the obstacle type and a corresponding system of transport equations.

To simplify the presentation, we use periodic boundary conditions. Thus, the spatial domain is the N-dimensional flat torus, $\mathbb{T}^N$. Our MFG is determined by a value function, $u : \mathbb{T}^N \to \mathbb{R}^d$, a probability density, $\theta : \mathbb{T}^N \to (\mathbb{R}^+)^d$, and a switching current, ν, that together satisfy the following system of variational inequalities:

$$\max\left(H^i(Du^i, x) + u^i - g(\theta^i), \max_j \left(u^i - u^j - \psi^{ij}\right)\right) = 0 \tag{1.1}$$

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coupled with the system

$$-\operatorname{div}(D_pH^i(Du^i,x)\theta^i)+\theta^i+\sum_{j\neq i}\left(\nu^{ij}-\nu^{ji}\right)=1. \tag{1.2}$$

Moreover, for $1\leq i,j\leq d$, ν^{ij} is a non-negative measure on $\mathbb{T}^N$ supported in the set $u^i-u^j-\psi^{ij}=0$.

This system models a stationary population of agents. Each agent moves in $\mathbb{T}^N$ and can switch between different modes that are given by the index i. Their actions seek to minimize a certain cost. Agents can change their state by continuously modifying their spatial position, x, and by switching between different modes, i to j, at a cost ψ^{ji}. The function $u^i(x)$ is the value function for an agent whose spatial location is x and whose mode is i. The function $\theta^i(x)$ is the density of the agents on $\mathbb{T}^N\times\{1,\dots,d\}$. Thus, we require that $\theta^i(x)\geq 0$. We note that θ^i is not a probability measure on $\mathbb{T}^N\times\{1,\dots,d\}$ because the source term in the right-hand side of (1.2) is not normalized.

In Section 2, we discuss detailed assumptions on the Hamiltonians H^i, on the nonlinearity g, and on the switching costs ψ^{ij}. A concrete example that satisfies those is

$$H^i(x,p)=\frac{|p|^2}{2}+V^i(x),\qquad g(\theta)=\ln\theta,\quad\text{and}\quad \psi^{ij}(x)=\eta, \tag{1.3}$$

with $V^i:\mathbb{T}^N\to\mathbb{R}$ being a C^∞ function and η being a positive real number. Another case of interest is the polynomial nonlinearity, $g(m)=m^\alpha$ for $\alpha>0$.

Standard MFGs involve two equations, a Hamilton–Jacobi equation and a transport or Fokker–Planck equation. This latter equation is the adjoint of the linearization of the former. Because the non-linear operator in (1.1) is non-differentiable, (1.2) is obtained by a limiting procedure. In the context of MFGs, this method was first used in [22]. Here, we consider the following penalized problem.

$$H^i(Du^i,x)+u^i+\sum_{j\neq i}\beta_\epsilon(u^i-u^j-\psi^{ij})=g(\theta^i) \tag{1.4}$$

$$-\operatorname{div}(D_pH^i(Du^i,x)\theta^i)+\theta^i+\sum_{j\neq i}\beta'_\epsilon(u^i-u^j-\psi^{ij})\theta^i-\beta'_\epsilon(u^j-u^i-\psi^{ji})\theta^j=1, \tag{1.5}$$

where the penalty function, β_ϵ, is an increasing C^∞ function and $\epsilon>0$. We assume that, as $\epsilon\to0$, $\beta_\epsilon(s)\to\infty$ for $s>0$ and $\beta_\epsilon(s)=0$ for $s\leq0$, see Assumption 8. The study of optimal switching has a long history that predates viscosity solutions and, certainly, MFGs, see, for example [2,5,6,16]. In those references, the use of a penalty to approximate a non-smooth Hamilton–Jacobi equation is a fundamental tool. The penalty in (1.4) is similar to the ones in the aforementioned references.

More recently, several authors have investigated weakly coupled Hamilton–Jacobi equations [44], the corresponding extension of the weak KAM and Aubry-Mather theories [4,13,40], the asymptotic behavior of solutions [3,17,42,41,45], and homogenization [43]. In these references, the state of the system has different modes, and a random process drives the switching between them. In contrast, here, the switches occur at deterministic times. Thus, our models are the MFG counterpart of the Hamilton–Jacobi systems considered in [18,32]. MFGs with different populations [11,12] are a limit case of (1.1)–(1.2). This can be seen by taking the limit $\psi^{ij}\to+\infty$; that is, the case where agents are not allowed to change their state.

The development of the existence and regularity theory for MFGs has seen substantial progress in recent years. Uniformly elliptic and parabolic MFGs are now well understood, and the existence of smooth and weak solutions has been established in a broad range of problems, see, respectively, [23,25,24,30,29,26, 27,10,46,47]. However, the regularity theory for first-order MFGs is less developed and, in general, only weak solutions are known to exist [7–9]. Variational inequality methods are at the heart of a new class of techniques to establish the existence of weak solutions, both for first- and second-order problems [19] and for their numerical approximation [1]. Some MFGs that arise in applications, such as congestion [20,33] or