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The first initial–boundary value problem for Hessian equations of parabolic type on Riemannian manifolds

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ABSTRACT

In this paper, we are concerned with the first initial–boundary value problem for a class of fully nonlinear parabolic equations on Riemannian manifolds. As usual, the establishment of the *a priori* C^2 estimates is our main part. Based on these estimates, the existence of classical solutions is proved under conditions which are nearly optimal.

1. Introduction

In this paper, we study the Hessian equations of parabolic type of the form

$$f(\lambda(\nabla^2 u + \chi), -u_t) = \psi(x, t) \tag{1.1}$$

in $M_T = M \times (0, T] \subset M \times \mathbb{R}$ satisfying the boundary condition

$$u = \varphi, \quad \text{on } \mathcal{P}M_T, \tag{1.2}$$

where (M, g) is a compact Riemannian manifold of dimension $n \geq 2$ with smooth boundary ∂M and $\overline{M} := M \cup \partial M$, $\mathcal{P}M_T = BM_T \cup SM_T$ is the parabolic boundary of M_T with $BM_T = M \times \{0\}$ and $SM_T = \partial M \times [0, T]$, f is a symmetric smooth function of $n+1$ variables, $\nabla^2 u$ denotes the Hessian of $u(x,t)$ with respect to $x \in M$, $u_t = \frac{\partial u}{\partial t}$ is the derivative of $u(x,t)$ with respect to $t \in [0, T]$, χ is a smooth (0, 2) tensor on $\bar{M}$ and $\lambda(\nabla^2 u + \chi) = (\widehat{\lambda}_1, \ldots, \widehat{\lambda}_n)$ denotes the eigenvalues of $\nabla^2 u + \chi$ with respect to the metric g.

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We assume f to be defined in an open convex cone $\Gamma \subset \mathbb{R}^{n+1}$ with vertex at the origin satisfying

$$\Gamma_{n+1} \equiv \{\lambda \in \mathbb{R}^{n+1} : \text{ each component } \lambda_i > 0,\ 1 \le i \le n+1\} \subseteq \Gamma \neq \mathbb{R}^{n+1}$$

and furthermore, Γ is invariant under interchange of any two λ_i, i.e. it is symmetric.

In this work, f is assumed to satisfy the following structural conditions as in [3] (see [9] also):

$$f_i \equiv \frac{\partial f}{\partial \lambda_i} > 0 \text{ in } \Gamma, \quad 1 \le i \le n+1, \tag{1.3}$$

$$f \text{ is concave in } \Gamma \tag{1.4}$$

and

$$\delta_{\psi,f} \equiv \inf_{M_T} \psi - \sup_{\partial\Gamma} f > 0, \quad \text{where } \sup_{\partial\Gamma} f \equiv \sup_{\lambda_0 \in \partial\Gamma} \limsup_{\lambda \to \lambda_0} f(\lambda). \tag{1.5}$$

In this work we are interested in the existence of classical solutions to (1.1)–(1.2). Recent research on the Hessian equations of elliptic type (see [9,7]):

$$f(\lambda(\nabla^2 u + \chi)) = \psi(x) \tag{1.6}$$

provides some ideas to deal with our Eq. (1.1) under nearly minimal restrictions on f.

The most typical examples of f satisfying (1.3)–(1.5) are $f = \sigma_k^{1/k}$ and $f = (\sigma_k/\sigma_l)^{1/(k-l)}$, $1 \le l < k \le n+1$, defined in the Gårding cone

$$\Gamma_k = \{\lambda \in \mathbb{R}^{n+1} : \sigma_j(\lambda) > 0,\ j = 1, \ldots, k\},$$

where σ_k are the elementary symmetric functions

$$\sigma_k(\lambda) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \ldots \lambda_{i_k}, \quad k = 1, \ldots, n+1.$$

When $f = \sigma_{n+1}^{1/(n+1)}$, Eq. (1.1) can be written as the parabolic Monge–Ampère equation:

$$-u_t \det(\nabla^2 u + \chi) = \psi^{n+1}, \tag{1.7}$$

which was introduced by Krylov in [19] when $\chi = 0$ in Euclidean space. Instead of the determinant in (1.7), Ren [25] studied equations of the form

$$-u_t f(\lambda(\nabla^2 u)) = \psi(x,t). \tag{1.8}$$

Our interest to study (1.1) is from their natural connection to the deformation of surfaces by some curvature functions. For example, Eq. (1.7) plays a key role in the study of contraction of surfaces by Gauss–Kronecker curvature (see Firey [5] and Tso [28]). For the study of more general curvature flows, the reader is referred to [1,2,14,24] and their references. (1.7) is also relevant to a maximum principle for parabolic equations (see Tso [29]).

In [23], Lieberman studied the first initial–boundary value problem of Eq. (1.1) when $\chi \equiv 0$ and ψ may depend on u and ∇u in a bounded domain $\Omega \subset \mathbb{R}^{n+1}$ under various conditions. Jiao and Sui [18] considered parabolic Hessian equations of the form

$$f(\lambda(\nabla^2 u + \chi)) - u_t = \psi(x,t) \tag{1.9}$$

on Riemannian manifolds under an additional condition which was introduced in [10]

$$T_\lambda \cap \partial\Gamma^\sigma \text{ is a nonempty compact set, } \forall \lambda \in \Gamma \text{ and } \sup_{\partial\Gamma} f < \sigma < f(\lambda), \tag{1.10}$$

where $\partial\Gamma^\sigma = \{\lambda \in \Gamma : f(\lambda) = \sigma\}$ is the boundary of $\Gamma^\sigma = \{\lambda \in \Gamma : f(\lambda) > \sigma\}$ and T_λ denote the tangent plane at λ of $\partial\Gamma^{f(\lambda)}$, for $\sigma > \sup_{\partial\Gamma} f$ and $\lambda \in \Gamma$. Eq. (1.9) in domains of $\mathbb{R}^n$ was also studied by Ivochkina