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On the convergence of solutions of inclusions containing maximal monotone and generalized pseudomonotone mappings

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ABSTRACT

We are concerned in this paper with the existence, boundedness, and the convergence of solutions to a sequence of inclusions

$$A_k(u) + B_k(u) \ni L_k,$$

where A_k is a maximal monotone mapping, B_k is a generalized pseudomonotone mapping defined on a reflexive Banach space X, and $L_k \in X^*$. We study appropriate kinds of convergence for A_k and B_k such that a limit of a sequence of solutions of these inclusions is also a solution of the limit inclusion.

1. Introduction

We are concerned in this paper with the existence, boundedness, and convergence of solutions to inclusions of the form

$$A(u) + B(u) \ni L, \tag{1.1}$$

in the case where A, B, and L all vary, and also with different types of convergence for multivalued mappings related to such inclusions. Here, A is a maximal monotone mapping and B is a generalized pseudomonotone mapping from a reflexive Banach space X into 2^{X^*} (X^* is the dual space of X), and $L \in X^*$. More specifically, we consider a sequence of inclusions

$$A_k(u) + B_k(u) \ni L_k, \tag{1.2}$$

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and study appropriate kinds of convergence for A_k, B_k such that a limit of a sequence of solutions of (1.2) is also a solution of the limit inclusion.

The convergence of solutions of nonsmooth problems seems to be first studied in the pioneering works [20,21], in which variational inequalities with convex functionals and (single-valued) monotone mappings were considered. A variational inequality with a convex functional ϕ can be written equivalently as an inclusion containing the maximal monotone mapping $\partial\phi$, where $\partial\phi$ is the subdifferential of ϕ. It was introduced in these papers the important concepts of convergence of convex sets and convex functionals (now called Mosco convergences, which later contributed to the general theory of Γ-convergence). The convergence in the graph sense of monotone mappings was also considered in [21]. Related extensions for the convergence of variational inequalities and their solutions are given in [1–4,10,19,22] and the references therein.

The existence of solutions of inclusions of the form (1.1) has been studied in e.g. [11,13–15]. The present paper is a continuation and extension of our previous work in [15]. Regarding the convergence of solutions of (1.2), a standard convergence for L_k is the (norm) convergence in X^*. For the maximal monotone mapping A_k, it was shown in Theorem 7.3 of [14] that when $B_k = B$ is fixed, a suitable kind of convergence for A_k is the convergence in the graph sense (cf. Condition (2.1) in [21], Definition 7.2 in [14], or Definition 2.1(b) in Section 3). A natural question arising from this theorem is that, if the perturbing generalized pseudomonotone term B_k also varies, what an appropriate type of convergence for B_k should be such that we still have convergence/continuous dependence of solutions of (1.2). For this purpose, we introduce a kind of convergence for multivalued mappings that extends the graph sense convergence for maximal monotone mappings. This convergence, called the convergence in the generalized pseudomonotone sense (or (gpm)-convergence for short, see Definition 2.1), comes out naturally from the concept of generalized pseudomonotone mappings introduced in the classical work [8]. We show (cf. Theorem 2.3 and Corollary 2.5) that under certain normalization condition, the convergence in the graph sense and that in the generalized pseudomonotone sense are in fact equivalent in the class of maximal monotone mappings.

For the existence and boundedness of solutions of (1.2), we slightly extend an existence result in [15] and show that under certain coercivity condition, the solutions of (1.2) are partially bounded, i.e., there exists a bounded sequence of solutions of (1.2). For the convergence of solutions of (1.2), we show in Theorem 3.2 that if $L_k \to L$ in X^* and $A_k + B_k \to A + B$ in the generalized pseudomonotone sense then a weak limit of a sequence of solutions of (1.2) is a solution of the limit inclusion (1.1). This result can also be seen as an existence result for (1.1).

On the other hand, this convergence–existence theorem leads, as in the theory of generalized pseudomonotone mappings, to the interesting question of obtaining the (gpm)-convergence of a sum sequence $\{A_k + B_k\}$ from the convergence of the component sequences $\{A_k\}$ and $\{B_k\}$. We show (cf. Theorem 4.2) that together with certain boundedness conditions, the sum of sequences convergent in the generalized pseudomonotone sense also converges in that sense. We also show in Theorem 5.2 that if one sequence in the sum converges in a stronger sense (called the $(S)_+$-convergence) and the other sequence satisfies certain compactness condition then the sum sequence is also (gpm)-convergent.

As illustrating examples of the various types of convergence introduced here, we consider multivalued mappings appearing in some variational and quasi-variational inequalities studied recently in [9,16–18]. The mappings A_k and B_k in our examples are given (formally) by the integrals

$$\langle A_k(u), v\rangle = \int_\Omega a_k(x, \nabla u)\nabla v dx, \tag{1.3}$$

and

$$\langle B_k(u), v\rangle = \int_\Omega b_k(x, u) v dx, \tag{1.4}$$