Nonlinear Analysis 143 (2016) 89–104

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Nonexistence of global solutions to critical semilinear wave equations in exterior domain in high dimensions

Ning-An Lai[a,*], Yi Zhou[b]

[a] Department of Mathematics, Lishui University, Lishui City 323000, China
[b] School of Mathematical Sciences, Fudan University, Shanghai 200433, China

ARTICLE INFO

Article history:
Received 26 January 2016
Accepted 13 May 2016
Communicated by Enzo Mitidieri

MSC:
35L05
35L70

Keywords:
Blow up
Critical semilinear wave equations
Exterior problem

ABSTRACT

The aim of this paper is to study the long time behavior of solutions to the initial boundary value problem of critical semilinear wave equations with small data in exterior domain in high dimensions ($n \geq 5$). We prove that solutions cannot exist globally in time. The novelty is that we first use a cutoff function to transform the exterior problem to a Cauchy problem. Then we divide the time into two intervals, in one time interval we establish an energy estimate to control the error terms while in the other interval the asymptotic behavior of two test functions is used. Furthermore, we obtain the upper bound of the lifespan by following the idea of Zhou and Han (2014).

1. Introduction

Let $n \geq 5$ and $\Delta = \sum_{i=1}^{n} \partial_{x_i}^2$ be the Laplace operator. We are going to consider the initial boundary value problem of critical semilinear wave equations in exterior domain

$$\begin{cases} u_{tt} - \Delta u = |u|^p, \quad t > 0,\ x \in \Omega^c \subseteq \mathbb{R}^n, \\ u(0,x) = \varepsilon f(x), \qquad u_t(0,x) = \varepsilon g(x), \quad x \in \Omega^c \subseteq \mathbb{R}^n, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1}$$

where $p = p_c(n)$ is the positive root of the quadratic equation (3) stated below. Ω^c denotes the exterior domain to $\Omega \subset \{x | x \in \mathbb{R}^n, |x| \leq \frac{1}{2}\}$, and ε represents the smallness of the data.

Problem (1) can date back to the well-known Strauss conjecture, which concerns the global and nonglobal existence of solutions to the Cauchy problem of semilinear wave equation with compact supported data

$$u_{tt} - \Delta u = |u|^p, \quad (t,x) \in \mathbf{R}_+ \times \mathbf{R}^n. \tag{2}$$

* Corresponding author. Tel.: +86 05782271250.
E-mail addresses: hyayue@gmail.com (N.-A. Lai), yizhou@fudan.edu.cn (Y. Zhou).

http://dx.doi.org/10.1016/j.na.2016.05.010
0362-546X/

The pioneering work to problem (2) with small data is [9], in which John proved that the Cauchy problem in $\mathbb{R}^3$ admits a critical power $p = 1 + \sqrt{2}$ in the sense that: when $1 < p < 1 + \sqrt{2}$ the solution blows up in a finite time while for $p > 1 + \sqrt{2}$ there exists global solution. After that, Strauss [21] conjectured that for each $n \geq 2$ there exists a critical power $p_c(n)$ for problem (2), which is the positive root of the quadratic equation

$$(n-1)p^2 - (n+1)p - 2 = 0. \tag{3}$$

And then many mathematicians have contributed to this conjecture. We refer the reader to the works of Glassey [4,5], Sideris [18], Rammaha [16], Jiao and Zhou [8], Schaeffer [17], Zhou [32], Lindblad and Sogge [15], Georgiev, Lindblad and Sogge [3], Tataru [24], Yordanov and Zhang [27], Zhou [33] and the authors [10].

It is also a long history for the study of the lifespan estimate to the Cauchy problem (2) both from above and below in the blow up case. And there is extensive literature on the question, see, for instance [10,13,23,22,26,29,31,30] and references therein.

Recently, it has attracted much attention to generalize Strauss conjecture to manifolds such as asymptotically Euclidean non-trapping Riemannian manifolds, Schwarzschild and Kerr spacetime. We refer the reader to [1] for blow up result and to [20,14,25] for global existence result.

It is also interesting to think about the corresponding problem outside obstacle, that is: the initial boundary value problem in exterior domain. We expect that it admits the same critical exponent as that of the Cauchy problem, based on some known results. Du et al. [2] proved global existence for $p > p_c(4)$ and then Hidano et al. [7] obtained the same result for $p > p_c(n)$ and $n = 3, 4$ (see also Yu [28] for trapping obstacles case). The case of $p > p_c(2)$ is due to the work of Smith, Sogge and Wang [19]. In the opposite direction, Zhou and Han [34] established blow up result and the upper bound of lifespan in the case $1 < p < p_c(n)$ and $n \geq 3$. Yu [28] gave the obstacle version of the sharp lifespan for semilinear wave equations when $p < p_c(3)$. Not much later, Li and Wang [12] (see also [6]) showed blow up for $1 < p < p_c(2)$. Very recently, the authors [11] studied the critical case $p = p_c(3)$ and proved the nonexistence of global solution. To our best knowledge, this is the only known result for the initial boundary value problem of critical semilinear wave equations in exterior domain so far.

This paper aims to study the exterior problem (1) with critical power $p = p_c(n)$ in high dimensions ($n \geq 5$). Blow up result and the upper bound of lifespan will be established, no matter how small the data are. The critical Cauchy problem in $\mathbb{R}^n$ in high dimensions ($n \geq 4$) has been solved by Yordanov and Zhang [27] and Zhou [33] by the test function method independently. The key ingredients in [27] are a test function and Radon transformation, while in [33] the key element is another special test function which solves the corresponding linear wave equation and is positive, homogeneous of degree $q (q > 0)$ and radial symmetric. However, both methods do not work for the corresponding exterior problem, since we have neither the Radon transformation nor the special test function as in [33]. For our purpose we use a cut off function to transform the initial boundary value problem into a corresponding Cauchy problem, then both the test functions introduced in [27,33] can be used. And we demonstrate the proof by two main steps: (a) establish the lower bound of the L^p normal of the solutions; (b) construct an auxiliary functional $G(t)$ and obtain the lower bound of $G'(t)$. To finish step (a), we divide the time into two intervals: $t \leq \delta\varepsilon^{-\alpha}$ (denoted by I) and $t > \delta\varepsilon^{-\alpha}$ (denoted by II). In the interval I an energy estimate (see Lemma 2.1) is used to control the error terms while in II we use the asymptotic behavior of the test function $F(x) = \int_{\mathbb{S}^{n-1}} e^{x\cdot\omega} d\omega$. For step (b), in the interval I we control the error terms by the same way as that in step (a) while in II we use the asymptotic properties of another test function $\Phi_q(t, x)$ (see Section 4).

Remark 1.1. Unfortunately, our method stated above fails to the initial boundary value problem of critical semilinear wave equation in exterior domain in 4-D, since we require sufficient decay of $\Phi_q(t, x)$, which is critically not enough in 4-D.