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Generalized Orlicz spaces and related PDE

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ABSTRACT

We prove the boundedness of the maximal operator in generalized Orlicz spaces defined on subsets of $\mathbb{R}^n$. The proof is based on an extension result for Φ-functions. We study generalized Sobolev–Orlicz spaces and establish density of smooth functions and the Poincaré inequality. As applications we establish the existence of solutions of the φ-Laplace equation with zero and non-zero right-hand side. Further, we systematize assumptions for Φ-functions and prove several basic tools needed for the study of differential equations of generalized Orlicz growth.

1. Introduction

Generalized Orlicz spaces $L^{\varphi(\cdot)}$ have been studied since the 1940s. A major synthesis of functional analysis in these spaces is given in the 1983-monograph of Musielak [26] hence the alternative name Musielak–Orlicz spaces. These spaces are similar to Orlicz spaces, but defined by a more general function $\varphi(x,t)$ which may vary with the location in space: the norm is defined by means of the integral

$$\int_{\mathbb{R}^n} \varphi(x, |f(x)|)\, dx,$$

whereas in an Orlicz spaces φ would be independent of $x, \varphi(|f(x)|)$. When $\varphi(t) = t^p$ we obtain classical Lebesgue spaces L^p.

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The special case $\varphi(x,t) := t^{p(x)}$, so-called variable exponent spaces $L^{p(\cdot)}$, and corresponding differential equations with non-standard growth have been vigorously studied in recent years [9,10,16]. The reason that variable exponent spaces thrived while little was done in generalized Orlicz spaces was the belief that many classical results can be obtained in the former setting but not the latter. However, this belief has been challenged recently, based on new techniques that were developed and perfected in the context of variable exponent spaces [11,14,18,19,22–24,27].

In addition to being a natural generalization which covers results from both variable exponent and Orlicz spaces, the study of generalized Orlicz spaces can be motivated by applications to image processing [1,5,15], fluid dynamics [31] and differential equations.

Regarding regularity theory of differential equations, Giannetti and Passarelli di Napoli [12] and Ok [28] as well as Baroni, Colombo and Mingione [2,3,6–8] studied the minimization problems

$$\min_u \int |\nabla u|^{p(x)} \log(e + |\nabla u|)\, dx \quad \text{and} \quad \min_u \int |\nabla u|^p + a(x)\, |\nabla u|^q\, dx,$$

respectively. They showed that the regularity of the minimizer depends on the regularity of the exponent p and the weight a. Both are special cases of generalized Orlicz growth.

In the function space setting the first steps from $L^{p(\cdot)}$ were similarly Φ-functions of type $t^{p(\cdot)} \log(e+t)^{q(\cdot)}$ like that of Giannetti, Passarelli di Napoli and Ok. Such Φ-functions were studied e.g. in [20,25]. To our mind, a better approach is to develop stronger tools and to move directly to study general Φ-functions including, among others, those studied by Colombo and Mingione. Our assumptions (A0)–(A2) have been shown to correspond to known optimal conditions in these special cases [18].

In this article we systematize assumptions for Φ-functions and prove several basic tools needed for the study of differential equations of generalized Orlicz growth (Sections 2–4). In order to use function spaces in subsets of $\mathbb{R}^n$, we study extensions of Φ-functions (Proposition 5.2). We consider generalized Sobolev–Orlicz spaces, including density of smooth functions (Theorem 6.6) and Poincaré inequality (Theorem 6.11). With these tools, we are able to establish the existence of solutions of the φ-Laplace equation with zero and non-zero right-hand side (Sections 7 and 8).

2. Φ-functions

The notation $f \lesssim g$ means that there exists a constant $C > 0$ such that $f \leqslant Cg$. The notation $f \approx g$ means that $f \lesssim g \lesssim f$. A function f is *almost increasing* if there exists a constant $L \geqslant 1$ such that $f(s) \leqslant Lf(t)$ for all $s \leqslant t$. *Almost decreasing* is defined similarly.

Let us start with looking at the foundations of Φ-functions. Different researchers have used different conditions, which we try to capture in the next definition.

Definition 2.1. We consider increasing functions $\varphi\colon [0,\infty) \to [0,\infty]$ with $\varphi(0) = \lim_{t\to 0^+} \varphi(t) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$. Such φ is called a *Φ-prefunction*. Furthermore, φ is called

(1) a *weak Φ-function* if, additionally, $t \mapsto \frac{\varphi(t)}{t}$ is almost increasing on $(0,\infty)$;
(2) a *Φ-function* if, additionally, it is left-continuous and convex;
(3) a *strong Φ-function* if, additionally, it is continuous in $\overline{\mathbb{R}}$ and convex.

While Φ-functions are most commonly used, the class of weak Φ-functions is good in the sense that it is invariant under equivalence. Strong Φ-functions, on the other hand, are nice when it comes to dealing with the inverse, see the end of this section.