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Conformal geometry of timelike curves in the $(1+2)$-Einstein universe

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ABSTRACT

We study the conformal geometry of timelike curves in the $(1+2)$-Einstein universe, the conformal compactification of Minkowski 3-space defined as the quotient of the null cone of $\mathbb{R}^{2,3}$ by the action by positive scalar multiplications. The purpose is to describe local and global conformal invariants of timelike curves and to address the question of existence and properties of closed trajectories for the conformal strain functional. Some relations between the conformal geometry of timelike curves and the geometry of knots and links in the 3-sphere are discussed.
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1. Introduction

Conformal Lorentzian geometry has played an important role in general relativity since the work of H. Weyl [49]. In the 1980s, it has been at the basis of the development of twistor approach to gravity by Penrose and Rindler [39] and it is one of the main ingredients in the recently proposed cyclic cosmological models in general relativity [37,38,47]. It also plays a role in the regularization of the Kepler problem [19,24,25,28], in conformal field theory [45], and in Lie sphere geometry [4,6]. For what concerns in particular the geometry

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of curves, while the subject of conformal Lorentzian invariants of null curves has received some attention [4,3,48], that of timelike curves seems to have been little studied before.

In this paper we investigate the geometry of timelike curves in $\mathcal{E}^{1,2}$, the conformal compactification of Minkowski 3-space defined as the space of oriented null lines of $\mathbb{R}^{2,3}$ through the origin. This study is intended as a preliminary step to understand the 4-dimensional case, which is that of physical interest. Despite some formal similarities, there are substantial differences between the conformal Riemannian and Lorentzian case: the Lorentzian space $\mathcal{E}^{1,2}$ has the topology of $S^1 \times S^2$, which is not simply connected; the global Lorentzian metrics on $\mathcal{E}^{1,2}$ are never maximally symmetric; the universal covering of the conformal group of $\mathcal{E}^{1,2}$ does not admit finite dimensional representations and has a center which is discrete, but not finite [2,42]. Following [3,14], we call $\mathcal{E}^{1,2}$ the $(1+2)$-Einstein universe.[1] A motivation for this terminology is that the universal covering of $\mathcal{E}^{1,2}$, $\mathbb{R} \times S^2$, endowed with the product metric $-dt^2 + g_{S^2}$, provides a static solution of Einstein's equation with a positive cosmological constant. This solution was proposed by Einstein himself as a model of a closed static universe in [9]. In addition, the pseudo-Riemannian geometries of the standard Friedmann–Lemaitre–Robertson–Walker cosmological models can be realized as subgeometries of $\mathcal{E}^{1,2}$ [20].

The purposes of this paper are threefold. The first is to describe local and global conformal differential invariants of a timelike curve. The second purpose is to address the question of existence and properties of closed trajectories for the variational problem defined by the conformal strain functional, the Lorentzian analogue of the conformal arclength functional in Möbius geometry [23,29,30,33]. The Lagrangian of the strain functional depends on third-order jets and shares many similarities with the relativistic models for massless or massive particles based on higher-order action functionals, a topic which has been much studied over the past twenty years [13,17,21,31,32,36,34,35,40]. The last purpose is to establish a connection between the conformal geometry of timelike curves in the Einstein universe and the geometry of transversal knots in the unit 3-sphere.

From a physical point of view, the relevant objects are the lifts of timelike curves to the universal covering of $\mathcal{E}^{1,2}$ and their global conformal invariants. The compactified model has the advantage of having a matrix group as its restricted conformal group, which simplifies the use of the geometric methods based on the transformation group and eases the computational aspects.

The paper is organized as follows. In Section 2, we collect some background material about conformal Lorentzian geometry. For the geometry of the Einstein universe, we mainly follow [3].

In Section 3, we study the conformal geometry of timelike curves in the Einstein universe. We define the infinitesimal conformal strain, which is the Lorentzian analogue of the conformal arc element of a curve in S^3 [22,27,29,30,33], and the notion of a conformal vertex. An explicit description of curves all of whose points are vertices is given in Proposition 1. Next, we define the concept of osculating conformal cycle and give a geometric characterization of conformal vertices in terms of the analytic contact between the curve and its osculating cycle (Proposition 2). We then prove the existence of a canonical conformal frame field along a generic timelike curve (i.e., a timelike curve without vertices) and define the two conformal curvatures, which are the main local conformal invariants of a generic curve (Theorem 3). As a byproduct, some elementary consequences are derived (Propositions 4–6 and Corollary 7).

In Section 4, the canonical conformal frame is used to investigate generic timelike curves with constant conformal curvatures. We exhibit explicit parameterizations of such curves in terms of elementary functions (Theorems 8 and 9) and discuss their main geometric properties. These are the Lorentzian counterparts of analogous results for curves with constant conformal curvatures in S^3 [44,46].

In Section 5, we use the canonical frame to compute the Euler–Lagrange equations of the conformal strain functional (Theorem 10). Consequently we show that the conformal curvatures of the extrema can be expressed in terms of Jacobi's elliptic functions. As a byproduct, we show that the conformal equivalence

[1] Actually, $\mathcal{E}^{1,2}$ is the double covering of the space that in [3,14] is called Einstein universe.

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