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# Eigenvalues of the drifting Laplacian on complete noncompact Riemannian manifolds

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ABSTRACT

In this paper, we investigate eigenvalues of the eigenvalue problem with Dirichlet boundary condition of the drifting Laplacian on an $n$-dimensional, complete noncompact Riemannian manifold. Some estimates for eigenvalues are obtained. By utilizing Cheng and Yang recursion formula, we give a sharp upper bound of the $k$th eigenvalue. As we know, product Riemannian manifolds, Ricci solitons and self-shrinkers are some important Riemannian manifolds. Therefore, we investigate the eigenvalues of the drifting Laplacian on those Riemannian manifolds. In particular, by some theorems of classification for Ricci solitons, we can obtain some eigenvalue inequalities of drifting Laplacian on the Ricci solitons with certain conditions.


## 1. Introduction

Let $(M^n, g)$ be an $n$-dimensional Riemannian manifold with a smooth metric $g$, then we say that the triple $(M^n, g, e^{-f}dv)$ is a metric measure space, where $f$ is a smooth function on $M^n$ and $e^{-f}dv$ is a weighted volume density on $M^n$. The $l$-dimensional Bakry–Emery Ricci curvature corresponding to weighted metric measure spaces is a very important curvature quantity, which is defined by

$$\mathrm{Ric}^f_{l,n} := \mathrm{Ric} + \mathrm{Hess} f - \frac{\nabla f \otimes \nabla f}{l-n},$$

where Ric and Hess$f$ denote Ricci tensor of $M^n$ and Hessian of $f$, respectively (see [4–6,24]). When $l$ is infinite, then the $l$-dimensional Bakry–Emery Ricci curvature becomes the $\infty$-dimensional Bakry–Emery Ricci curvature

$$\mathrm{Ric}^f := \lim_{l\to\infty} \mathrm{Ric}^f_{l,n} = \mathrm{Ric} + \mathrm{Hess} f.$$

Furthermore, let $f$ be a constant in the above equation, we have

$$\mathrm{Ric}^f = \mathrm{Ric}.$$

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Therefore, the Bakry–Émery Ricci tensor is naturally viewed as an extension of the Ricci tensor. On the metric measure space, we define the drifting Laplacian (or called $f$-Laplacian, Witten–Laplacian) as follows:

$$\Delta_f u := \Delta u - \langle \nabla f, \nabla u \rangle = e^f \operatorname{div}\left(e^{-f} \nabla u\right),$$

where $\Delta$ is Laplacian on the Riemannian manifold $M^n$. It is easy to show that the drifting Laplacian is a self-adjoint operator in the sense of the weighted measure $e^{-f}dv$, and the drifting Laplacian is exactly the Laplacian when $f$ is a constant. Drifting Laplacian is an important elliptic operator and widely applied in the probability theory and geometrical analysis. In particular, many mathematicians pay more and more attention to the estimates for eigenvalues of the drifting Laplacian in the last two decades. On one hand, upper bounds for the first eigenvalue of the drifting Laplacian on complete Riemannian manifolds have been studied in [27,28,33,34]. On the other hand, many mathematicians have also considered the lower bounds for the Eigenvalues of the drifting Laplacian on the compact Riemannian manifolds. Du [26] and Li and Wei [24], for example, have studied the Reilly formula with respect to the complete smooth measure space to obtain a lower bound of the first eigenvalue for the drifting Laplacian on the $f$-minimal hypersurface. Furthermore, they have given a Lichnerowicz type lower bound for the first eigenvalue of the drifting Laplacian on the compact manifolds with positive Bakry–Émery Ricci curvature. In addition, we refer the readers to [1,18,17] and the reference therein for the further interesting results.

Assume that $\Omega \subset M^n$ is a bounded domain with piecewise smooth boundary $\partial\Omega$ in an $n$-dimensional complete metric measure space $(M^n, g, d\mu)$. Let $\lambda_i$ be the $i$th eigenvalue of the Dirichlet eigenvalue problem of drifting Laplacian:

$$\begin{cases} \Delta_f u = -\lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

It is well known that the spectrum of the Dirichlet problem (1.1) is discrete and satisfies the following:

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \leq \cdots \to +\infty,$$

where each eigenvalue is repeated according to its multiplicity.

When $M^n$ is an $n$-dimensional Euclidean space $\mathbb{R}^n$, Payne, Pólya and Weinberger [29,30] studied the universal inequalities for eigenvalues of the Dirichlet problem (1.1) of Laplacian. They obtained

$$\lambda_{k+1} - \lambda_k \leq \frac{4}{nk} \sum_{i=1}^{k} \lambda_i. \tag{1.2}$$

In fact, many mathematicians extended the universal inequality of Payne, Pólya and Weinberger in some differential backgrounds. However, there are two main contributions due to Hile and Protter [22] and Yang [36]. In 1980, Hile and Protter proved the following universal inequality of eigenvalues:

$$\sum_{i=1}^{k} \frac{\lambda_i}{\lambda_{k+1} - \lambda_i} \geq \frac{nk}{4}, \tag{1.3}$$

which is sharper than the result of Payne, Pólya and Weinberger. Furthermore, Yang [36] (cf. [13]) obtained a very sharp universal inequality:

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \lambda_i. \tag{1.4}$$

From (1.4), we yield

$$\lambda_{k+1} \leq \frac{1}{k} \left(1 + \frac{4}{n}\right) \sum_{i=1}^{k} \lambda_i. \tag{1.5}$$