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Maximal function characterizations of variable Hardy spaces associated with non-negative self-adjoint operators satisfying Gaussian estimates☆

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ABSTRACT

Let $p(\cdot):\ \mathbb{R}^n \to (0,1]$ be a variable exponent function satisfying the globally log-Hölder continuous condition and L a non-negative self-adjoint operator on $L^2(\mathbb{R}^n)$ whose heat kernels satisfying the Gaussian upper bound estimates. Let $H_L^{p(\cdot)}(\mathbb{R}^n)$ be the variable exponent Hardy space defined via the Lusin area function associated with the heat kernels $\{e^{-t^2L}\}_{t\in(0,\infty)}$. In this article, the authors first establish the atomic characterization of $H_L^{p(\cdot)}(\mathbb{R}^n)$; using this, the authors then obtain its non-tangential maximal function characterization which, when $p(\cdot)$ is a constant in $(0,1]$, coincides with a recent result by L. Song and L. Yan (2016) and further induces the radial maximal function characterization of $H_L^{p(\cdot)}(\mathbb{R}^n)$ under an additional assumption that the heat kernels of L have the Hölder regularity.

1. Introduction

The main purpose of this article is to establish the non-tangential or radial maximal function characterizations of the Hardy space $H_L^{p(\cdot)}(\mathbb{R}^n)$ introduced in [49]. Recall that the theory of classical Hardy spaces on the Euclidean space $\mathbb{R}^n$ was introduced and developed in the 1960s and 1970s. Precisely, the real-variable theory of Hardy spaces on $\mathbb{R}^n$ was initiated by Stein and Weiss [43] and then systematically developed by Fefferman and Stein [24], which has played an important role in modern harmonic analysis

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and been widely used in partial differential equations (see, for example, [12,24,42]). As is well known, the classical Hardy space is intimately connected with the Laplace operator $\Delta := -\sum_{i=1}^{n} \partial_{x_i}^2$ on $\mathbb{R}^n$. Indeed, for $p \in (0,1]$, the Hardy space $H^p(\mathbb{R}^n)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ (the set of all *tempered distributions*) such that the area integral function

$$S(f)(\cdot) := \left\{ \int_0^\infty \int_{|y-\cdot|<t} \left| t^2 \Delta e^{-t^2\Delta}(f)(y) \right|^2 \frac{dy\,dt}{t^{n+1}} \right\}^{\frac{1}{2}}$$

belongs to $L^p(\mathbb{R}^n)$. Moreover, for $p \in (0,1]$, the Hardy space $H^p(\mathbb{R}^n)$ involves several different equivalent characterizations, for example, if $f \in \mathcal{S}'(\mathbb{R}^n)$, then

$$\begin{aligned} f \in H^p(\mathbb{R}^n) &\Longleftrightarrow \sup_{t\in(0,\infty)} \left| e^{-t^2\Delta}(f) \right| \in L^p(\mathbb{R}^n) \\ &\Longleftrightarrow \sup_{t\in(0,\infty),\, |y-\cdot|<t} \left| e^{-t^2\Delta}(f)(y) \right| \in L^p(\mathbb{R}^n). \end{aligned}$$

Also, it is well known that the Hardy space $H^p(\mathbb{R}^n)$, with $p \in (0,1]$, is a suitable substitute of the Lebesgue space $L^p(\mathbb{R}^n)$, for example, the classical Riesz transform is bounded on $H^p(\mathbb{R}^n)$, but not on $L^p(\mathbb{R}^n)$ when $p \in (0,1]$. However, in many situations, the standard theory of Hardy spaces is not applicable, for example, the Riesz transform $\nabla L^{-1/2}$ may not be bounded from the Hardy space $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ when L is a second-order divergence form elliptic operator with complex bounded measurable coefficients (see [29]). Motivated by this, the topic for developing a real-variable theory of Hardy spaces that are adapted to different differential operators has inspired great interests in the last decade and has become a very active research topic in harmonic analysis (see, for example, [3,6,21–23,28,29,31,33,45,46,49]).

Particularly, let L be a linear operator on $L^2(\mathbb{R}^n)$ and generate an analytic semigroup $\{e^{-tL}\}_{t>0}$ with heat kernels having pointwise upper bounds. Then, by using the Lusin area function associated with these heat kernels, Auscher, Duong and McIntosh [3] initially studied the Hardy space $H^1_L(\mathbb{R}^n)$ associated with the operator L. Based on this, Duong and Yan [21,22] introduced the BMO-type space $\mathrm{BMO}_L(\mathbb{R}^n)$ associated with L and proved that the dual space of $H^1_L(\mathbb{R}^n)$ is just $\mathrm{BMO}_{L^*}(\mathbb{R}^n)$, where L^* denotes the *adjoint operator* of L in $L^2(\mathbb{R}^n)$. Later, Yan [45] further generalized these results to the Hardy spaces $H^p_L(\mathbb{R}^n)$ with p close to, but less than, 1 and, more generally, the Orlicz–Hardy space associated with such operator was investigated by Jiang et al. [33]. Very recently, under the assumption that L is a non-negative self-adjoint operator whose heat kernels satisfying Gaussian upper bound estimates, Song and Yan [41] established a characterization of Hardy spaces $H^p_L(\mathbb{R}^n)$ via the non-tangential maximal function associated with the heat semigroup of L based on a subtle modification of technique due to Calderón [8], which was further generalized into the Musielak–Orlicz–Hardy space in [47].

Another research direction of generalized Hardy spaces is the variable exponent Hardy space, which also extends the variable Lebesgue space. Recall that the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$, with a variable exponent $p(\cdot): \mathbb{R}^n \to (0,\infty)$, consists of all measurable functions f such that $\int_{\mathbb{R}^n} |f(x)|^{p(x)}\,dx < \infty$. The study of variable Lebesgue spaces can be traced back to Birnbaum–Orlicz [5] and Orlicz [37], but the modern development started with the article [34] of Kováčik and Rákosník as well as [13] of Cruz-Uribe and [17] of Diening, and nowadays have been widely used in harmonic analysis (see, for example, [14,18]). Moreover, variable function spaces also have interesting applications in fluid dynamics [1,38], image processing [10], partial differential equations and variational calculus [2,27,39]. Recall that the variable exponent Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ was introduced by Nakai and Sawano [36] and, independently, by Cruz-Uribe and Wang [16] with some weaker assumptions on $p(\cdot)$ than those used in [36], which was further investigated by Sawano [40], Zhuo et al. [51] and Yang et al. [50].

Let $p(\cdot): \mathbb{R}^n \to (0,1]$ be a variable exponent function satisfying the globally log-Hölder continuous condition. Very recently, the authors [49] introduced the Hardy space $H^{p(\cdot)}_L(\mathbb{R}^n)$ via the Lusin area function