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Parabolicity of minimal graphs in Riemannian warped products and rigidity theorems

Juan A. Aledo[a,*], Rafael M. Rubio[b]

[a] *Departamento de Matemáticas, E.S.I. Informática, Universidad de Castilla-La Mancha, 02071 Albacete, Spain*

[b] *Departamento de Matemáticas, Campus de Rabanales, Universidad de Córdoba, 14071 Córdoba, Spain*

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ABSTRACT

We provide natural geometric assumption for a minimal graph in a Riemannian warped product to be parabolic. As a consequence, we deal with several Bernstein-type problems.

1. Introduction

In this paper we deal with the following nonlinear elliptic equation in divergence form

$$\operatorname{div}\left(\frac{Du}{f(u)\sqrt{f(u)^2+|Du|^2}}\right)=\frac{f'(u)}{\sqrt{f(u)^2+|Du|^2}}\left\{2-\frac{|Du|^2}{f(u)^2}\right\}, \qquad (1)$$

where $f:\mathbb{R}\to(0,+\infty)$ is a positive smooth function and the unknown function u takes values on a complete Riemannian surface (M,g_M). Here, D and div denote the gradient and the divergence operators for (M,g_M), respectively.

* Corresponding author.
E-mail addresses: juanangel.aledo@uclm.es (J.A. Aledo), rmrubio@uco.es (R.M. Rubio).

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As is well-known, $u : M \to \mathbb{R}$ satisfies (1) if it is extremal among functions under interior variation for the area integral

$$\int_M f(u)\sqrt{f(u)^2 + \mid Du \mid^2}. \tag{2}$$

This variational problem naturally arises in Riemannian Geometry. In fact, consider the standard metric dt^2 on the real line $\mathbb{R}$ and the product manifold $\overline{M} = \mathbb{R} \times M$ endowed with the Riemannian metric

$$\bar{g} = \pi_{\mathbb{R}}^*(dt^2) + f(\pi_{\mathbb{R}})^2\, \pi_M^*(g_M), \tag{3}$$

where $\pi_{\mathbb{R}}$ and π_M are the projections onto $\mathbb{R}$ and M, respectively. Then, $(\overline{M}, \bar{g})$ is a Riemannian warped product which will be denoted by $\overline{M} = \mathbb{R} \times_f M$.

Given a domain Ω in M and $u \in C^\infty(\Omega)$, the induced metric on Ω via the graph $\Sigma_u = \{(u(p), p) :\ p \in \Omega\}$ is given by

$$g_u = du^2 + f(u)^2 g_M.$$

Then u is a critical point of (2) if and only if the associated graph Σ_u has zero mean curvature. Namely, (1) is the minimal surfaces equation on M for graphs in $\overline{M}$. When $f \equiv 1$, (1) reduces to

$$\operatorname{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right) = 0. \tag{4}$$

Moreover, when M is the Euclidean plane, (4) becomes the well-known minimal surfaces equation in $\mathbb{R}^3$, and the celebrated classical Bernstein's Theorem states that the only entire solutions $u \in \mathcal{C}^\infty(\mathbb{R}^2)$ to this equation are the affine planes. In particular, the only entire *bounded* solutions to the minimal surfaces equation on $\mathbb{R}^2$ are the constant functions.

Regarding the general equation (1), if there exists $t_0 \in \mathbb{R}$ such that $f'(t_0) = 0$, then the constant function $u = t_0$ is an entire solution to (1). Thus, the following questions arise in a natural way:

When is the constant function $u = t_0$ with $f'(t_0) = 0$ the unique entire solution to Eq. (1)?

When is there no entire solution to Eq. (1)?

The main aim of this paper is to give several answers to these questions under suitable geometrical assumptions.

Next we revise some previous works regarding this subject. Entire minimal graphs in Riemannian products $\mathbb{R} \times M$, with M a Riemannian surface, have been studied by several authors. When M is a complete surface with non-negative Gaussian curvature, Rosenberg [11] proved that an entire minimal graph in $\mathbb{R} \times M$ is totally geodesic. It follows since such a minimal graph is stable, because vertical translations are isometries of $\mathbb{R} \times M$ (see [10]) and every complete stable minimal surface in a 3-manifold with non-negative Ricci curvature is totally geodesic (Schoen, [12]).

More recently, Alías, Dajczer and Ripoll have obtained the same result using a different technique (see [2, Th. 4]). Specifically, these authors prove the stability of the entire minimal graph showing that the cosine of the angle between the unitary normal vector field on the graph and the coordinate vector field on the first factor is a positive Jacobi field, together with a well-known result due to Fischer-Colbrie and Schoen, which ensures that the graph must be totally geodesic [4].

Our approach is completely different to those aforementioned: on one hand, it allows to extend the study to the case of Riemannian warped products, and on the other hand the assumption on the Gaussian curvature can be weakened. In order to do that, we take a suitable metric pointwise conformal to the induced metric on the minimal graph from $\overline{M}$, which lets a certain control of its Gaussian curvature (see Lemma 2). Indeed, the graph endowed with this conformal metric has finite total curvature when the fiber is a complete