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Entropy solutions for a traffic model with phase transitions

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ABSTRACT

We introduce an appropriate notion of entropy solution for a generalization of the two phase macroscopic traffic model proposed in Goatin (2006). We first apply the wave-front tracking method to prove existence and a priori bounds for weak solutions. Then, in the case the characteristic field of the free phase is linearly degenerate, we prove that the obtained weak solutions are in fact entropy solutions. The case of solutions attaining values at the vacuum is considered. We also present an explicit numerical example to describe some qualitative features of the solutions.

1. Introduction

This paper deals with macroscopic modelling of traffic flows. The existing literature on macroscopic models for traffic flows is already vast and characterized by contributions motivated by their real life applications, as the surveys [5,24,25,28] and the books [13,27] demonstrate.

The macroscopic variables that translate the discrete nature of traffic into continuous variables are the velocity v, namely the space covered per unit time by the vehicles, the density ρ, namely the number of vehicles per unit length of the road, and the flow f, namely the number of vehicles per unit time. By definition we have that

$$f = \rho\, v. \tag{1}$$

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Clearly, the macroscopic variables are in general functions of time $t > 0$ and space $x \in \mathbb{R}$. By imposing the conservation of the number of vehicles along a road with no entrances or exits we deduce the scalar conservation law

$$\rho_t + f_x = 0. \tag{2}$$

Since the system (1), (2) has three unknown variables, a further condition has to be imposed. There are two main approaches to do it. First order macroscopic models close the system (1), (2) by giving beforehand an explicit expression of one of the three unknown variables in terms of the remaining two. The prototype of the first order models is the Lighthill, Whitham [21] and Richards [26] model (LWR). The basic assumption of LWR is that the velocity of any driver depends on the density alone, namely

$$v = V(\rho).$$

The function $V\colon [0, \rho_{\max}] \to [0, v_{\max}]$ is given beforehand and is assumed to be $\mathbf{C^1}$, non-increasing, with $V(0) = v_{\max}$ and $V(\rho_{\max}) = 0$, where $\rho_{\max}$ is the maximal density corresponding to the situation in which the vehicles are bumper to bumper, and $v_{\max}$ is the maximal speed corresponding to the free road. As a result, LWR is given by the scalar conservation law

$$\rho_t + [\rho\, V(\rho)]_x = 0.$$

Second order macroscopic models close the system (1), (2) by adding a further conservation law. The most celebrated second order macroscopic model is the Aw, Rascle [4] and Zhang [29] model (ARZ). Away from the vacuum, ARZ writes

$$\rho_t + [\rho\, v]_x = 0, \qquad [\rho\,(v + P(\rho))]_t + [\rho\,(v + P(\rho))\, v]_x = 0,$$

where the "pressure" function $P(\rho)$ plays the role of an anticipation factor, taking into account drivers' reactions to the state of traffic in front of them.

The main drawback of LWR is the unrealistic behaviour of the drivers, who take into account the slightest change in the density and adjust instantaneously their velocities according to the densities they are experiencing. Moreover, experimental data show that the fundamental diagram (ρ, f) is given by a cloud of points rather than being the support of a map $\rho \mapsto [\rho\, v(\rho)]$. ARZ can be interpreted as a generalization of LWR, possessing a family of fundamental diagram curves, rather than a single one. For this reason ARZ avoids the drawbacks of LWR listed above. Moreover, traffic hysteresis, which means that for the same distance headway drivers choose a different speed during acceleration from that chosen during deceleration, can be reproduced with ARZ but not with LWR.

On the other hand, the system describing ARZ degenerates into just one equation at the vacuum $\rho = 0$. In particular, as pointed out in [4], the solutions to ARZ fail to depend continuously on the initial data in any neighbourhood of $\rho = 0$; moreover, as observed in [16], the solutions may experience a sudden increase of the total variation as the vacuum appears.

For the above reasons, Goatin [15] proposes to couple ARZ with LWR by introducing a two phase transition model. More precisely, the phase transition model proposed in [15] describes the dynamics in the free flow and those in the congested flow respectively with LWR and ARZ. In fact, this allows to better fit the experimental data, and has also the advantage of correcting the exposed drawbacks of LWR in the congested traffic and of ARZ at the vacuum.

In [14] the authors point out that the model proposed in [15] does not satisfy properties that they consider necessary to model appropriately urban road networks, namely:

- Vehicles stop only at maximum density, i.e. the velocity v is zero if and only if the density ρ is equal to the maximum density possible $\rho_{\max}$.
- The density at a red traffic light is the maximum possible, i.e. $\rho_{\max}$.