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$C^{1,\alpha}$ estimates for the parallel refractor

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ABSTRACT

We consider the parallel refractor problem when the planar radiating source lies in a medium having higher refractive index than the medium in which the target is located. We prove local $C^{1,\alpha}$ estimates for parallel refractors under suitable geometric assumptions on the source and target, and under local regularity hypotheses on the target set. We also discuss existence of refractors under energy conservation assumptions.

1. Introduction

Suppose we have a domain $\Omega \subset \mathbb{R}^n$ and a domain Σ contained in an n dimensional surface in $\mathbb{R}^{n+1}$; here, Ω denotes the extended source, and Σ denotes the target domain, receiver, or screen to be illuminated. Let n_1 and n_2 be the indices of refraction of two homogeneous and isotropic media I and II, respectively. Suppose from the extended source Ω, surrounded by medium I, radiation emanates in the vertical direction e_{n+1} with intensity $f(x)$ for $x \in \Omega$, and the target Σ is surrounded by medium II. That is, all emanating rays from Ω are collimated. A *parallel refractor* is an optical surface $\mathcal{R}$, interface between media I and II, such that all rays refracted by $\mathcal{R}$ into medium II are received at the surface Σ with prescribed radiation intensity $\sigma(p)$ at each point $p \in \Sigma$. Assuming no loss of energy in this process, we have the conservation of energy equation $\int_\Omega f(x)\,dx = \int_\Sigma \sigma(p)\,dp$.

When medium II is denser than medium I (i.e. $n_1 < n_2$), $C^{1,\alpha}$ estimates are proved in [12], and the existence of refractors is proved in [11]. The purpose of this paper is to consider the case when $n_1 > n_2$. This has interest in the applications to lens design since lenses are typically made of a material having a refractive index larger than the surrounding medium. In fact, if the material around the source is cut out with a plane parallel to the source, then the lens sandwiched between that plane and the constructed refractor surface

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will perform the desired refracting job. When $n_1 > n_2$ the geometry of the refractors is different than when $n_1 < n_2$; in fact, the geometry is determined by hyperboloids instead of ellipsoids. In addition, in case $n_1 > n_2$, total internal reflection can occur and one needs additional geometric conditions on the relative configuration between the source and the target so that the target is reachable by the refracted rays. To obtain existence and regularity of refractors when $n_1 > n_2$, the use of hyperboloids requires non-trivial changes in some of the arguments used in [12] when $n_1 < n_2$. The main differences are in the set up of the problem, in the arguments to obtain global support from local support, Section 4, and in the proof of existence. Our results are local; that is, we only need to assume local conditions in a neighborhood of a point in the extended source and the target. The main result of the paper is Theorem 5.4 where $C^{1,\alpha}$ estimates are proved. We remark that most results do not involve the energy distribution given in the source and target, and conservation of energy is only used to prove existence in Theorem A.1. For instance, the fact that local refractors are global, Theorem 4.2, just follows from the geometric assumptions in Section 3; see condition (AW). In addition, Theorem 5.3 only requires geometric assumptions. Properties of the target measure are necessary only to obtain the Hölder estimates, Theorem 5.4. Our results are structural, in the sense that they only depend on the geometric conditions assumed and do not depend on the smoothness of the measures given in the source and target.

Problems of refraction have generated interest recently for the applications to design free form lenses and also for the various mathematical tools developed to solve them. For example, the far field point source refractor problem is solved in [8] using mass transport. The near field point source refractor problem is considered in [7,9]. More general models taking into account losses due to internal reflection are in [10]. Numerical methods have been developed in [2,4] for the actual calculation of reflectors, and recently in [5] for the numerical design of far field point source refractors. A significant amount of work has also been done to obtain results on the regularity of reflectors and refractors [3,18,15,13,14,6].

The organization of the paper is as follows. Section 2 contains results concerning estimates of hyperboloids of revolution. The precise definition of refractor is in Section 2.2, and the structural assumptions on the target that avoid total reflection are in Section 2.3. The derivative estimates needed for hyperboloids are in Section 2.4. Section 3 contains assumptions on the target modeled on the conditions introduced by Loeper in the seminal work [17, Proposition 5.1]. In Section 4, using the geometry of the hyperboloids, we prove that if a hyperboloid supports a parallel refractor locally, then it supports the refractor globally provided the target satisfies the local condition (AW). This resembles the condition (A3) of Ma, Trudinger and Wang [19] introduced in the context of optimal mass transport. The main results are in Section 5; in particular, Section 5.1 contains the proof of the Hölder estimates. Finally, in the Appendix, we discuss and establish the existence of refractors satisfying the energy conservation condition (A.17).

2. Definitions and preliminary results

We briefly review the process of refraction. Points in $\mathbb{R}^{n+1}$ will be denoted by $X = (x, x_{n+1})$. We consider parallel rays traveling in the unit direction e_{n+1}. Let T be a hyperplane with outward pointing unit normal N and $X \in T$. We assume that medium I is located in the region below T and media II in the region above T. In such a scenario, a ray of light emanated from Ω in the direction e_{n+1} strikes T at X and, by Snell's Law of Refraction, gets refracted in the unit direction

$$\Lambda = \kappa\, e_{n+1} + \delta N, \quad \text{with } \delta = -\kappa\, e_{n+1} \cdot N + \sqrt{1 + \kappa^2\, \left((e_{n+1} \cdot N)^2 - 1\right)},$$

where $\delta < 0$ since $\kappa = n_1/n_2 > 1$. The refracted ray is $X + s\Lambda$, for $s > 0$; see Fig. 1. In particular, if $v \in \mathbb{R}^n$ and the hyperplane T is so that the unit upper normal $N = \frac{(-v,1)}{\sqrt{1+|v|^2}}$, then $\delta = \frac{-\kappa+\sqrt{1-(\kappa^2-1)|v|^2}}{\sqrt{1+|v|^2}}$ and the