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Sharp estimates of unimodular multipliers on frequency decomposition spaces

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A B S T R A C T

This paper is devoted to give a general method for the study of unimodular multipliers on certain function spaces defined by decomposition on the frequency plane. These spaces include modulation spaces, α-modulation spaces and (homogeneous and inhomogeneous) Besov spaces. We give a complete characterization of the Fourier multipliers on these function spaces, and also give a characterization of unimodular multipliers under some mild assumptions. As applications, we obtain some sharp boundedness properties of unimodular Fourier multipliers between these function spaces. We also obtain the asymptotic estimates for certain free dispersive semigroups.

1. Introduction

One of the core problems in harmonic analysis is to study the boundedness of certain linear operators on various function spaces or distribution spaces. These operators are usually raised from the researches on partial differential equation (PDE), mathematical physics, probability theory and other fields. For instance, the Fourier multiplier operator $e^{it|\Delta|^{\beta/2}}$, with the Laplace operator Δ, is the fundamental semi-group of the Schrödinger equation when $\beta = 2$ and is the fundamental semi-group of the wave equation when $\beta = 1$. On the other hand, choosing a right function space (or distribution space) as a work frame is a crucial step in order to obtain the well-posedness of certain Cauchy or boundary value problems of partial differential

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equations. Thus, an important work is to study various function spaces and distribution spaces to fit different PDE problems. To this end, one of the most efficient methods of defining function spaces is to use the frequency decomposition of $\mathbb{R}^n$. In the Euclidean space $\mathbb{R}^n$, a family of countable subsets $\mathcal{Q} = \{Q_i\}_{i\in I}$ is called an admissible covering (see [10]) of $\mathbb{R}^n$ if

(1) $\mathbb{R}^n = \bigcup_{i\in I} Q_i$;
(2) $\sup_{i\in I} \#(\{j \in I : \ Q_i \cap Q_j \neq \emptyset\}) < \infty$.

Let χ_{Q_i} be the characterization function of Q_i and $\widetilde{\chi}_{Q_i}$ be a smooth modification of χ_{Q_i} so that it is a smooth bump functions related to Q_i for each $i \in I$. For distributions f, consider the frequency projection

$$P_{Q_i}(f) = \widetilde{\chi}_{Q_i}\mathscr{F}(f).$$

Here $\mathscr{F}$ is the Fourier transform with the inverse Fourier transform $\mathscr{F}^{-1}$. Choose an appropriate sequence of positive numbers $\{\lambda_j\}$. For $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, we can define the norm

$$\|f\|_{\Gamma^s_{p,q}} = \left\| \left\{ \sum_{j\in I} \left(\lambda_j^s \left|\mathscr{F}^{-1}P_{Q_j}(f)\right|\right)^q \right\}^{1/q} \right\|_{L^p}.$$

The space $\Gamma^s_{p,q}$ then is the set of all f satisfying $\|f\|_{\Gamma^s_{p,q}} < \infty$. When we let Q_j be the dyadic regions $\left\{\xi \in \mathbb{R}^n : 2^j \le |\xi| < 2^{j+1}\right\}$ and $\{\lambda_j\} = \{2^j\}$, the space $\Gamma^s_{p,q}$ becomes the homogeneous Triebel–Lizorkin space that includes the classical Lebesgue space L^p and the Sobolev space $W^{p,s}$ if one chooses appropriate parameters s and q. Related to the Triebel–Lizorkin space is the famous Littlewood–Paley theory that plays a remarkable role in the study of harmonic analysis and PDE.

Also, we can define the frequency decomposition space

$$\Phi^s_{p,q} = \left\{ f : \|f\|_{\Phi^s_{p,q}} = \left\{ \sum_{j\in I} \left(\lambda_j^s \left\|\mathscr{F}^{-1}P_{Q_j}(f)\right\|_{L^p}\right)^q \right\}^{1/q} < \infty \right\}.$$

$\Phi^s_{p,q}$ becomes the modulation space $M^s_{p,q}$ if we choose the uniform unit cubes $\{Q_j : j \in \mathbb{Z}^n\}$ in the frequency space and $\lambda_j = (1 + |j|^2)^{\frac{1}{2}}$ and $\Phi^s_{p,q}$ becomes the Besov space $B^s_{p,q}$ if we choose a dyadic decomposition on the frequency space. Besides $M^s_{p,q}$ and $B^s_{p,q}$, the α-modulation spaces $M^{s,\alpha}_{p,q}$, $0 \le \alpha < 1$, can be derived from $\Phi^s_{p,q}$ by choosing a sequence of α coverings (see Section 2).

In this article, our main interest will be on the spaces $M^s_{p,q}$, $B^s_{p,q}$ and $M^{s,\alpha}_{p,q}$. We will give the precise definitions of these spaces in Section 2. The Besov space is a well known function space. The reader can easily find a lot of research articles on the Besov space, through the Google Scholar. Below, we briefly review some historical developments of the modulation spaces $M^s_{p,q}$ and α-modulation spaces $M^{s,\alpha}_{p,q}$.

The modulation space $M^s_{p,q}$ was originally introduced by Feichtinger [8] in 1983 by the short-time Fourier transform. It can be used to measure the size and smoothness of a function in a way different from the L^p space and the Sobolev space. Nowadays, people discover that this space has a discrete version based on the uniform unit decomposition on the frequency space. Based on this alternative definition, many notable performances were showed on the modulation space when people study PDE and pseudo-differential operators. For the research work on the modulation space, among numerous references, the reader can see [8,12,22] for many elementary properties of modulation space, [2,1,3] for the study of boundedness on modulation spaces for certain operators, [4,22,21,23] for the study of nonlinear evolution equations related to modulation space.

As we mentioned above, the inhomogeneous Besov space $B^s_{p,q}$ is another frequency-decomposition function space based on the dyadic decomposition. Thus, it is interesting to build a bridge connecting the modulation