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Existence and symmetry of solutions for critical fractional Schrödinger equations with bounded potentials

Xia Zhang[a], Binlin Zhang[b,*], Dušan Repovš[c]

[a] *Department of Mathematics, Harbin Institute of Technology, Harbin 150001, PR China*
[b] *Department of Mathematics, Heilongjiang Institute of Technology, Harbin 150050, PR China*
[c] *Faculty of Education and Faculty of Mathematics and Physics, University of Ljubljana, Kardeljeva ploščad 16 SI-1000 Ljubljana, Slovenia*

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ABSTRACT

This paper is concerned with the following fractional Schrödinger equations involving critical exponents:

$$(-\Delta)^\alpha u + V(x)u = k(x)f(u) + \lambda|u|^{2^*_\alpha-2}u \quad \text{in } \mathbb{R}^N,$$

where $(-\Delta)^\alpha$ is the fractional Laplacian operator with $\alpha \in (0,1)$, $N \geq 2$, λ is a positive real parameter and $2^*_\alpha = 2N/(N-2\alpha)$ is the critical Sobolev exponent, $V(x)$ and $k(x)$ are positive and bounded functions satisfying some extra hypotheses. Based on the principle of concentration compactness in the fractional Sobolev space and the minimax arguments, we obtain the existence of a nontrivial radially symmetric weak solution for the above-mentioned equations without assuming the Ambrosetti–Rabinowitz condition on the subcritical nonlinearity.

1. Introduction and main result

In this paper, we study the solutions of the following Schrödinger equations involving a critical nonlinearity:

$$(-\Delta)^\alpha u + V(x)u = k(x)f(u) + \lambda|u|^{2^*_\alpha-2}u \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

driven by the fractional Laplacian operator $(-\Delta)^\alpha$ of order $\alpha \in (0,1)$, where $N \geq 2, \lambda$ is a positive real parameter and $2^*_\alpha = 2N/(N-2\alpha)$ is the critical Sobolev exponent.

* Corresponding author.
E-mail addresses: piecesummer1984@163.com (X. Zhang), zhangbinlin2012@163.com (B. Zhang), dusan.repovs@guest.arnes.si (D. Repovš).

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The fractional Laplacian operator $(-\Delta)^\alpha$, which (up to normalization constants), may be defined as

$$(-\Delta)^\alpha u(x) := \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2\alpha}}\, dy, \quad x \in \mathbb{R}^N,$$

where P.V. stands for the principal value. It may be viewed as the infinitesimal generators of a Lévy stable diffusion processes (see [1]). This operator arises in the description of various phenomena in the applied sciences, such as phase transitions, materials science, conservation laws, minimal surfaces, water waves, optimization, plasma physics and so on, see [13] and references therein for more detailed introduction. Here we would like to point out some interesting models involving the fractional Laplacian, such as, the fractional Schrödinger equation (see [14,15,22–24]), the fractional Kirchhoff equation (see [17,32,33,46,47]), the fractional porous medium equation (see [11,45]), the fractional Yamabe problem (see [34]) and so on, have attracted recently considerable attention. As a matter of fact, the literature on fractional operators and their applications to partially differential equations is quite large, here we would like to mention a few, see for instance [3,9,12,26,27,35,38].

In what follows, let us sketch the related advance involving the fractional Schrödinger equations with critical growth in recent years. In [41], Shang and Zhang studied the existence and multiplicity of solutions for the critical fractional Schrödinger equation:

$$\varepsilon^{2\alpha}(-\Delta)^\alpha u + V(x)u = \lambda f(u) + |u|^{2^*_\alpha-2}u \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

Based on variational methods, they showed that problem (1.2) has a nonnegative ground state solution for all sufficiently large λ and small ε. In this paper, the following monotone condition was imposed on the continuous subcritical nonlinearity f:

$$f(t)/t \text{ is strictly increasing in } (0,+\infty). \tag{1.3}$$

Observe that (1.3) implies $2F(t) < f(t)t$, where $F(t) := \int_0^t f(\xi)\, d\xi$. Moreover, Shen and Gao in [43] obtained the existence of nontrivial solutions for problem (1.2) under various assumptions on $f(t)$ and potential function $V(x)$, in which the authors assumed the well-known Ambrosetti–Rabinowitz condition ((AR) condition for short) on f:

$$\text{there exists } \mu > 2 \text{ such that } 0 < \mu F(t) \le f(t)t \quad \text{for any } t > 0. \tag{1.4}$$

See also recent papers [42,36] on the fractional Schrödinger equations (1.2). In [44], Teng and He were concerned with the following fractional Schrödinger equations involving a critical nonlinearity:

$$(-\Delta)^\alpha u + u = P(x)|u|^{p-2}u + Q(x)|u|^{2^*_\alpha-2}u \quad \text{in } \mathbb{R}^N \tag{1.5}$$

where $2 < p < 2^*_\alpha$, potential functions $P(x)$ and $Q(x)$ satisfy certain hypotheses. Using the s-harmonic extension technique of Caffarelli and Silvestre [10], the concentration-compactness principle of Lions [29] and methods of Brézis and Nirenberg [8], the author obtained the existence of ground state solutions. On fractional Kirchhoff problems involving critical nonlinearity, see for example [2,31] for some recent results. Last but not least, fractional elliptic problems with critical growth, in a bounded domain, have been studied by some authors in the last years, see [4,5,18,28,39,40] and references therein.

On the other hand, Feng in [16] investigated the following fractional Schrödinger equations:

$$(-\Delta)^\alpha u + V(x)u = \lambda|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \tag{1.6}$$

where $2 < p < 2^*_\alpha, V(x)$ is a positive continuous function. By using the fractional version of concentration compactness principle of Lions [29], the author obtained the existence of ground state solutions to problem (1.6) for some $\lambda > 0$. Zhang et al. in [48] considered the following fractional Schrödinger equations with a critical nonlinearity:

$$(-\Delta)^\alpha u + u = \lambda f(u) + |u|^{2^*_\alpha-2}u \quad \text{in } \mathbb{R}^N. \tag{1.7}$$