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Nonlocal problems for Boussinesq equations

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ABSTRACT

In this paper, the existence and uniqueness of solution of the integral boundary value problem for abstract Boussinesq equations are obtained.

1. Introduction

The subject of this paper is to study the local existence and uniqueness of solution of the nonlocal integral boundary value problem (NIBVP) for the Boussinesq-operator equation

$$u_{tt} - \Delta u_{tt} + Au = \Delta f(u), \quad x \in R^n,\ t \in (0,T), \tag{1.1}$$

$$u(0,x) = \varphi(x) + \int_0^T \alpha(\sigma)\, u(\sigma,x)\, d\sigma, \tag{1.2}$$

$$u_t(0,x) = \psi(x) + \int_0^T \beta(\sigma)\, u_t(\sigma,x)\, d\sigma,$$

where A is a linear operator in a Banach space E, $\alpha(s)$ and $\beta(s)$ are measurable functions on $(0,T)$, $u(x,t)$ denotes the E-valued unknown function, $f(u)$ is the given nonlinear function, $\varphi(x)$ and $\psi(x)$ are the given

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initial value functions, subscript t indicates the partial derivative with respect to t, n is the dimension of space variable x and Δ denotes the Laplace operator in R^n.

Remark 1.1. Note that, particularly, the conditions (1.2) can be expressed as the following multipoint nonlocal conditions

$$u(0,x) = \varphi(x) + \sum_{k=1}^{l} \alpha_k u(\lambda_k, x), \qquad u_t(0,x) = \psi(x) + \sum_{k=1}^{m} \beta_k u_t(\lambda_k, x),$$

where l is a positive integer, α_k, β_k are complex numbers and $\lambda_k \in (0,T]$.

We are interested in studying the stability of solutions of problem (1.1)–(1.2) under some assumption on $\alpha(s)$, $\beta(s)$ and $\varphi(x), \psi(x)$.

Since the Banach space E is arbitrary and A is a possible linear operator, by choosing E and A and integral conditions, we can obtain numerous classes of nonlocal boundary value problem for generalized Boussinesq type equations which occur in a wide variety of physical systems, particularly in the propagation of longitudinal deformation waves in an elastic rod, hydro-dynamical process in plasma, in materials science which describe spinodal decomposition, in the absence of mechanical stresses (see [24,48,51,14] and the references therein). For example, if we choose $E = L^q(\Omega)$, $\Omega \in R^m$, $D(A) = W_0^{2,q}(\Omega)$, $Au = -\Delta_y u$, $\alpha_k = 0$ and $\beta_k = 0$, we obtain the Cauchy problem for generalized Boussinesq type equation

$$u_{tt} - \Delta u_{tt} - \Delta_y u = \Delta f(u), \quad x \in R^n, \ t \in (0,T), \tag{1.3}$$

$$u(0,x,y) = \varphi(x,y), \qquad u_t(0,x,y) = \psi(x,y), \tag{1.4}$$

where

$$\Delta_y u = \sum_{k=1}^{m} \frac{\partial^2 u}{\partial y_k^2}, \quad u = u(t,x,y), \ y = (y_1, y_2, \ldots, y_m) \in \Omega.$$

Eq. (1.3) arises in different situations (see [24,48]). For example, Eq. (1.3) for $n = 1$ describes a limit of a one-dimensional nonlinear lattice [51], shallow-water waves [14,19] and the propagation of longitudinal deformation waves in an elastic rod [6]. Rosenau [29] derived the equations governing dynamics of one, two and three-dimensional lattices, one of those equations (in dimensionless variables) is Eq. (1.3). In [45,46] the existence of the global classical solutions and the blow-up of the solution for the initial boundary value problem and the Cauchy problem (1.3)–(1.4) are obtained.

Here, by inspiring [45,46], the Cauchy problem for Boussinesq-operator equation is considered. Note that, differential operator equations were studied e.g. in [10,12,15,40,8,3,41,9,26–28,2,1,4,7,13,16,21,23,22, 39,38,33,35,34,32,30,31,49,17,37,36,47,50]. Cauchy problems for abstract hyperbolic equations were treated e.g. in [12,15,40,8,3,41,9,26–28,4]. In this paper, we obtain the local existence and uniqueness of small-amplitude solution of the problem (1.1)–(1.2). The strategy is to express the Boussinesq equation as an integral equation, to treat in the nonlinearity as a small perturbation of the linear part of the equation, then use the contraction mapping theorem and utilize an estimate for solutions of the linearized version to obtain a priori estimates on L^p norms of solutions. The key step is the derivation of the uniform estimate for the solutions of the NIBVP for the linearized Boussinesq equation. Harmonic analysis, the method of operator theory, interpolation of Banach Spaces, embedding theorems in abstract Sobolev spaces are the main tools implemented to carry out the analysis.

In order to state our results precisely, we introduce some notations and some function spaces.

Definitions and background

Let E be a Banach space. $L^p(\Omega; E)$ denotes the space of strongly measurable E-valued functions that are defined on the measurable subset $\Omega \subset R^n$ with the norm