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A new approach to Sobolev spaces in metric measure spaces

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ABSTRACT

Let (X, d_X, μ) be a metric measure space where X is locally compact and separable and μ is a Borel regular measure such that $0 < \mu(B(x, r)) < \infty$ for every ball $B(x, r)$ with center $x \in X$ and radius $r > 0$. We define $\mathcal{X}$ to be the set of all positive, finite non-zero regular Borel measures with compact support in X which are dominated by μ, and $\mathcal{M} = \mathcal{X} \cup \{0\}$. By introducing a kind of mass transport metric $d_{\mathcal{M}}$ on this set we provide a new approach to first order Sobolev spaces on metric measure spaces, first by introducing such for functions $F : \mathcal{X} \to \mathbb{R}$, and then for functions $f : X \to [-\infty, \infty]$ by identifying them with the unique element $F_f : \mathcal{X} \to \mathbb{R}$ defined by the mean-value integral:

$$F_f(\eta) = \frac{1}{\|\eta\|} \int f \, d\eta.$$

In the final section we prove that the approach gives us the classical Sobolev spaces when we are working in open subsets of Euclidean space $\mathbb{R}^n$ with Lebesgue measure.

1. Introduction

Suppose (X, d_X, μ) is a metric measure space and $1 \leq p < \infty$. If we want to introduce a first order Sobolev-type space, analogous to the classical Sobolev spaces $H^{1,p}(X)$ when X is an open subset of $\mathbb{R}^n$, d_X the Euclidean distance and μ the Lebesgue measure, then there is by now a few approaches available, most notably that based on upper gradients, which were introduced by Heinonen and Koskela [9], such as first studied by Shanmugalingam in [12]. By now there are (at-least) two good books which treat this approach in detail, first [2] by Björn and Björn and very recently [10] by Heinonen, Koskela, Shanmugalingam and Tyson.

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Apart from the Newtonian spaces there are alternative definitions of Sobolev spaces on metric measure spaces worth mentioning. Early approaches are due to Hajłasz in [5] and Korevaar–Schoen (a version directly comparable to this article of the latter approach seems first to have been developed in [11]). Other approaches can be found in [3] by Cheeger and [13] by Shvartsman. There have also been some axiomatic treatments (see e.g. [4,14]). The survey articles [6,7] are also worth mentioning as well as the book [8] which treats weighted Sobolev spaces on $\mathbb{R}^n$.

The idea of upper gradients is based on the well-known formula

$$|u(\gamma(s)) - u(\gamma(0))| \leq \int_0^s g(\gamma(t))\, dt \tag{1}$$

which holds for every smooth function in $\mathbb{R}^n$ and every rectifiable curve γ parametrized by arc-length, in case we put $g = |\nabla u|$. In a metric space we do not have a direct substitute for ∇u, but one then says that a Borel measurable function g is an upper gradient of u if the above formula holds for all curves. In case $g \in L^p(X)$ one says that g is a p-upper gradient of u. If $u \in L^p(X)$ has an upper gradient which also belongs to $L^p(X)$, then one says that u belongs to the **Newtonian space** $N^{1,p}(X)$, and give it the norm

$$\|u\|_{N^{1,p}(X)} = \left(\int |u|^p\, d\mu + \inf_g \int g^p\, d\mu \right)^{1/p},$$

where the infimum is taken over all upper gradients g of u.

For many questions it is desirable to have a minimal upper gradient $\tilde{g}_u$ of u such that the above infimum is attained. As it turns out however such a minimal upper gradient does not always exist, and we are forced to introduce the somewhat technical concept of curve modulus to introduce weak upper gradients which satisfy inequality (1) for "almost every" curve, which is given a precise meaning thorough the concept of curve modulus. It turns out that there is a unique, as an element in L^p, p-weak upper gradient $\tilde{g}_u$ of u, if u has an upper gradient in L^p.

The aim of this paper is to look at an alternative approach. We do not know in general how these spaces are related to the Newtonian ones, but at the very least we do indeed get the classical Sobolev spaces in case X is an open subset of $\mathbb{R}^n$ with Lebesgue measure (which in turn are equivalent to the Newtonian spaces in this setting).

To outline the approach assume that (X, d_X, μ) is a metric measure space, where X is separable and locally compact, and $0 < \mu(B) < \infty$ for every ball $B \subset X$. Let $\mathcal{M}$ denote the set of all (non-negative Radon) measures on X which are dominated by μ and have compact support, and let $\mathcal{X} = \mathcal{M} \setminus \{0\}$. In Section 4 we introduce a metric $d_{\mathcal{M}}$ on the set $\mathcal{M}$, and we give $\mathcal{X}$ the induced metric. The idea is to first look at real-valued functions F on $\mathcal{X}$, and to relate functions f on X to such by the mean-value integral as follows. If $\eta \in \mathcal{X}$ and f is a locally integrable function on X, then we define

$$F_f(\eta) = \frac{1}{\|\eta\|} \int f\, d\eta,$$

where $\|\eta\| = \int d\eta$ is the total mass of η. It is worthwhile to remark that if f is a locally integrable function on X, then point values are not really well defined (in the sense that we may have several representatives which are equal almost everywhere), but the value of F_f on elements in $\mathcal{X}$ is always well defined and finite. So the elements of $\mathcal{X}$ have a similar role to test functions. This is perhaps the major motivation for this type of approach. In some sense L^p-functions are more natural to think of as certain type of functions on $\mathcal{X}$ rather than X, and hence it seems natural to see to what extent one can carry the calculus to this set in a natural way.

In Section 5 we introduce a norm $\|\cdot\|_{\mathcal{L}^p(\mathcal{X})}$ on the set of extended real-valued functions on $\mathcal{X}$, and we let $\mathcal{L}^p(\mathcal{X})$ denote the set of such functions for which this expression is finite. In case $f \in L^p(X)$, then