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An instability result in the theory of suspension bridges

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We consider a second order system of two ODEs which arises as a single mode Galerkin projection of the so-called fish-bone (Berchio and Gazzola, 2015) model of suspension bridges. The two unknowns represent flexural and torsional modes of vibration of the deck of the bridge. The elastic response of the cables is supposed to be asymptotically linear under traction, and asymptotically constant when compressed (a generalization of the slackening regime). We establish a condition depending on a set of 3 parameters under which the flexural motions are unstable provided the energy is sufficiently large.

1. Introduction

An important issue in the mathematical modeling of suspension bridges is the phenomenon of energy transfer from flexural to torsional modes of vibration along the deck of the bridge. The classical and most accepted explanation for the sudden appearance of large flexural or twisting vibrations is aeroelastic fluttering, that is a dynamic instability of the elastic structure of the bridge caused by positive feedback between the deflection of the deck of the bridge and the force exerted by the wind.

The purpose of this paper is to provide a contribution to a recent field of research [1,2,5,6,8,11–13] according to which, internal nonlinear resonances, which depend only on the structural properties of the bridge model, may occur even when the aeroelastic coupling is disregarded and, as a consequence, the external forces driven by the wind are not considered. In particular, the main inspiration of this paper comes from the work of F. Gazzola and coworkers [1,2,6] to which we refer for more references and motivations.

The suspension bridge model under consideration has been proposed by K.S. Moore [14], revisited and somehow simplified in [2], whose authors called it *fish-bone* model. It is a two degree of freedom system

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of PDEs in which the dynamics of the midline of the deck, modeled as an Euler–Bernoulli vibrating beam of length L and width $2l$, is coupled with the elastic response of the suspension cables acting on the side ends of the deck. The (for example rectangular) cross section of the deck is assumed to be a rigid rod with constant mass density ρ, length $2l$ and negligible thickness with respect to l. If we denote by $Y(x,t)$ the vertical downward deflection of the midline of the deck with respect to the unloaded state, and by $\Theta(x,t)$ the angle of rotation of the deck with respect to the horizontal position, the mathematical description of the model is provided by the following system:

$$\begin{cases} \rho S\, Y_{tt} + EI\, Y_{xxxx} + \mathcal{F}(Y + l\sin\Theta) + \mathcal{F}(Y - l\sin\Theta) = 0, & 0 < x < L, \\ \rho J\, \Theta_{tt} - GJ\Theta_{xx} + l\cos\Theta\,[\mathcal{F}(Y + l\sin\Theta) - \mathcal{F}(Y - l\sin\Theta)] = 0, & 0 < x < L, \end{cases} \tag{1}$$

with the hinged boundary conditions:

$$Y(0,t) = Y(L,t) = Y_{xx}(0,t) = Y_{xx}(L,t) = 0, \qquad \Theta(0,t) = \Theta(L,t) = 0. \tag{2}$$

About the meaning of the constant parameters not yet defined, S is the cross section area, I is the planar second moment of area with respect to the plane $Y = 0$, J is the polar second moment of area with respect to the x-axis (being ρJ the torsional moment of inertia of the x-section of the bridge) and E and G are respectively Young's modulus and the shear modulus (being EI the flexural rigidity and GJ the torsional rigidity). The force $\mathcal{F}$ represents the restoring action exerted by the hangers in addition to gravity, and is applied to both extremities of the deck whose displacements from the unloaded state are given by $Y \pm l\sin\Theta$.

A reasonable choice (see [2,14]), is to assume that the suspension cables do not resist compression, and behave as linear springs of elastic constant $\mathrm{k} > 0$ if stretched (here g is the gravity). In this case we have the following nonlinear response (slackening regime):

$$\mathcal{F}(r) = \mathrm{k}\left[(r + r_0)^+ - r_0\right], \quad r_0 = \rho S g/2\mathrm{k}. \tag{3}$$

Now, as usual, we make a first simplification on this model: we denote by $Z(x,t)$ the vertical displacement with respect to the midline of the right edge of the road at position x, and $f(r) = \frac{1}{\rho S}\mathcal{F}(r)$; if we assume that, at least at the beginning, Θ is small enough, we have $\sin\Theta \sim \Theta, \cos\Theta \sim 1$. Setting $Z = l\Theta$, the system (1) becomes

$$\begin{cases} Y_{tt} + \dfrac{EI}{\rho S} Y_{xxxx} + f(Y + Z) + f(Y - Z) = 0, & 0 < x < L,\ t > 0 \\ Z_{tt} - \dfrac{G}{\rho} Z_{xx} + \dfrac{Sl^2}{J}\,(f(Y + Z) - f(Y - Z)) = 0, & 0 < x < L,\ t > 0, \end{cases} \tag{4}$$

with the hinged boundary conditions:

$$Y(0,t) = Y(L,t) = Y_{xx}(0,t) = Y_{xx}(L,t) = 0, \qquad Z(0,t) = Z(L,t) = 0. \tag{5}$$

Actually the results that we present in this paper depend only on certain properties of the nonlinear function f, essentially the fact that it has a finite limit slope as the displacements at the side edges of the deck $Y \pm Z$ get large. For this reason, we slightly generalize the law (3), with the following:

Assumption (H). (a) f is an increasing, continuous function such that $f(0) = 0$;

(b) f is piecewise C^1, that is it has continuous derivative with the exception of a finite (eventually empty) set of points $r_1 < r_2 < \cdots < r_n$ not including zero in which there exist the finite limits:

$$\lim_{r \to r_i^\pm} f'(r);$$

(c) f has asymptotically constant slope as $r \to \infty$, more precisely there exist the limits:

$$\lim_{r \to +\infty} f'(r) = m > 0, \qquad \lim_{r \to -\infty} f'(r) = 0.$$