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Extension properties and boundary estimates for a fractional heat operator

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ABSTRACT

The square root of the heat operator $\sqrt{\partial_t - \Delta}$, can be realized as the Dirichlet to Neumann map of the heat extension of data on $\mathbb{R}^{n+1}$ to $\mathbb{R}^{n+2}_+$. In this note we obtain similar characterizations for general fractional powers of the heat operator, $(\partial_t - \Delta)^s$, $s \in (0, 1)$. Using the characterizations we derive properties and boundary estimates for parabolic integro-differential equations from purely local arguments in the extension problem.

1. Introduction

In recent years there has been a surge in the study of the fractional Laplacian $(-\Delta)^s$ as well as more general linear and non-linear fractional operators. From an applied perspective a natural parabolic extension of $(-\Delta)^s$ is the parabolic operator $\partial_t + (-\Delta)^s$ which appears, for example, in the study of stable processes and in option pricing models, see [4] and the references therein. An other generalization is the time-fractional diffusion equation $\partial_t^\beta + (-\Delta)^s$ being the sum of a fractional and non-local time-derivative as well as a non-local operator in space as well. This type of equations has attracted considerable interest during the last years, mostly due to their applications in the modeling of anomalous diffusion, see [1,18,19], and the references therein. Decisive progress in the study of the fine properties of solutions to $(-\Delta)^s u = 0$ has been achieved through an extension technique, rediscovered in [6], based on which the fractional Laplacian can be studied through a local but degenerate elliptic operator having degeneracy determined by an A_2-weight. The latter operators have been thoroughly studied in [5,12,10,11,24], as well as in several other subsequent papers. Due to the lack of an established extension technique for operators of the forms $\partial_t + (-\Delta)^s$, $\partial_t^\beta + (-\Delta)^s$, more

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modest, but still important, progress has been made concerning these equations, again see [4,1,18,19], and the references therein.

In this note we take a different approach by considering directly the fractional heat operator $(\partial_t - \Delta)^s$. Given $s \in (0,1)$ we introduce the fractional heat operator $(\partial_t - \Delta)^s$ defined on the Fourier transform side by multiplication with the multiplier

$$(|\xi|^2 - i\tau)^s.$$

Using [23] it follows that $(\partial_t - \Delta)^s$ can be realized as a parabolic hypersingular integral,

$$(\partial_t - \Delta)^s f(x,t) = \frac{1}{\Gamma(-s)} \int_{-\infty}^{t} \int_{\mathbb{R}^n} \frac{(f(x,t) - f(x',y'))}{(t-t')^{1+s}} W(x-x', t-t')\, dx'dt',$$

where $W(x,t) = (4\pi t)^{-n/2} \exp(-|x|^2/(4t))$ for $t > 0$ and where $\Gamma(-s)$ is the gamma function evaluated at $-s$. The main result established in this note is that, in analogy with [6], fine properties of solutions to $(\partial_t - \Delta)^s f = 0$ can be derived through an extension technique based on which the fractional heat operator can be studied through a local but degenerate parabolic operator having degeneracy determined by an A_2-weight. To be precise, we consider a specific extension to the upper half space

$$\mathbb{R}^{n+2}_+ = \{(X,t) = (x, x_{n+1}, t) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} :\ x_{n+1} > 0\},$$

having boundary

$$\mathbb{R}^{n+1} = \{(x, x_{n+1}, t) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} :\ x_{n+1} = 0\}.$$

In the following we let $\nabla = (\nabla_x, \partial_{x_{n+1}})$ an we let div be the associated divergence operator. Let $a = 1 - 2s$. Letting

$$\Gamma_{x_{n+1}}(x,t) := \frac{1}{4^s \Gamma(-s)} x_{n+1}^{1-a} \frac{1}{t^{1+s}} W(x,t) \exp(-|x_{n+1}|^2/(4t)) \tag{1.1}$$

whenever $(x, x_{n+1}, t) \in \mathbb{R}^{n+2}_+$ and $t > 0$, we introduce, given a and $f \in C_0^\infty(\mathbb{R}^{n+1})$, the function

$$u(X,t) = u(x, x_{n+1}, t) = \int_{-\infty}^{t} \int_{\mathbb{R}^n} f(x',t') \Gamma_{x_{n+1}}(x-x', t-t')\, dx'dt'. \tag{1.2}$$

Given $(x,t) \in \mathbb{R}^{n+1}$ and $r > 0$, let $B(x,r)$ denote the standard Euclidean ball and let $C_r(x,t)$ denote the standard parabolic cylinder

$$C_r(x,t) = B(x,r) \times (t - r^2, t + r^2).$$

Our first result is the following theorem.

Theorem 1. *Consider s, $0 < s < 1$, fixed and let $a = 1 - 2s$. Consider $f \in C_0^\infty(\mathbb{R}^{n+1})$ and let u be defined as in (1.2). Then u solves*

$$\begin{aligned} {x_{n+1}}^a \partial_t u(X,t) - \mathrm{div}({x_{n+1}}^a \nabla u(X,t)) = 0, \qquad (X,t) \in \mathbb{R}^{n+2}_+,\\ u(x,0,t) = f(x,t), \quad (x,t) \in \mathbb{R}^{n+1}, \end{aligned} \tag{1.3}$$

and

$${x_{n+1}}^a \partial_{x_{n+1}} u(X,t)\Big|_{x_{n+1}=0} = -\lim_{x_{n+1} \to 0} 4^s \frac{u(X,t) - u(x,0,t)}{{x_{n+1}}^{1-a}} = (\partial_t - \Delta)^s f(x,t).$$

Furthermore, assume that $(\partial_t - \Delta)^s f(x,t) = 0$ whenever $(x,t) \in C_r(\tilde{x}, \tilde{t})$, for some $(\tilde{x}, \tilde{t}) \in \mathbb{R}^{n+1}$, $r > 0$, let $\tilde{u}(x, x_{n+1}, t)$ be defined to equal $u(x, x_{n+1}, t)$ whenever $x_{n+1} \geq 0$ and defined to equal $u(x, -x_{n+1}, t)$ whenever $x_{n+1} < 0$. Then $\tilde{u}$ is a weak solution to the equation

$$|x_{n+1}|^a \partial_t \tilde{u}(X,t) - \mathrm{div}(|x_{n+1}|^a \nabla \tilde{u}(X,t)) = 0,$$

in $\{(X,t) = (x, x_{n+1}, t) \in \mathbb{R}^{n+2} : (x,t) \in C_r(\tilde{x}, \tilde{t}),\ x_{n+1} \in (-1,1)\}$.