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Monotonicity of the period of a non linear oscillator

Rafael D. Benguria[a,*], M. Cristina Depassier[a], Michael Loss[b]

[a] Instituto de Física, Pontificia Universidad Católica de Chile, Chile
[b] School of Mathematics, Georgia Institute of Technology, United States

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ABSTRACT

We revisit the problem of monotonicity of the period function for the differential equation $u'' - u + u^p = 0$ and give a simple proof of recent results of Miyamoto and Yagasaki.

1. Introduction

The differential equation

$$-\frac{d^2u}{dx^2} + \frac{\lambda}{p-1}(u - u^p) = 0, \tag{1}$$

despite its simple form, plays an interesting role in the theory of ordinary differential equations and in the calculus of variations. When considered on the whole line it is the Euler–Lagrange equation for the sharp constant for a one dimensional Gagliardo Nirenberg inequality and can be solved explicitly. (For this and related results see [5].) When considered on the circle, i.e., an interval with periodic boundary conditions, the equation exhibits interesting bifurcation behavior. More precisely, fix $p > 1$ and consider the problem of minimizing

$$\frac{\int_{\mathbb{S}^1} |u'(x)|^2 dx + \frac{\lambda}{p-1} \int_{\mathbb{S}^1} |u(x)|^2 dx}{\left(\int_{\mathbb{S}^1} |u(x)|^{p+1} dx\right)^{2/(p+1)}}. \tag{2}$$

It is easily seen that a minimizer exists and that it satisfies an equation of type (1). It can be shown that for $\lambda \leq 1$, $u =$ constant is the only solution, while for $\lambda > 1$ there is an additional non-constant solution (see, e.g., [6]). Let us mention that the same problem on the d-dimensional sphere was treated in [1,2], where

* Corresponding author.
E-mail addresses: rbenguri@fis.puc.cl (R.D. Benguria), mcdepass@fis.puc.cl (M. Cristina Depassier), loss@math.gatech.edu (M. Loss).

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it is shown that the constant solution is the only solution for $\lambda \leq d$. For $\lambda > d$ the minimizing solution is not constant and not much is known about it.

One way to understand the result in one dimension is through the period of the solutions. If λ is small one expects that the period of any non constant solution is large compared to the circumference of the circle and hence the circle can only support the constant solution. If λ increases one expects that the period decreases and hence at some point a second solution bifurcates that is periodic and non-constant but fully supported by the circle. This new solution is then the minimizer of the functional.

Such problems can be very effectively studied using the period function. This idea, to our knowledge, goes back to a paper by Smoller and Wasserman [10] who use the period function to derive the complete bifurcation diagram for such type of equations with cubic non-linearities. Schaaf [9] proved the monotonicity of the period as a function of the energy for a class of Hamiltonian systems. This work was extended by Rothe [8]. Chow and Wang [4] developed alternative formulas that allowed them to prove the monotonicity of the period function for equations of the type

$$u'' + e^u - 1 = 0.$$

Some of the results of Chow and Wang were also obtained by Chicone [3] using a different approach. For further references on the uses of the period function the reader may consult [11].

It is somewhat surprising that, despite it ubiquity, the monotonicity of the period function for problem (1) in full generality was only established recently. Chicone's work was the starting point for a thorough investigation of (1) by Miyamoto and Yagasaki [7] who proved the monotonicity of the period function of (1) for integer p. This result was then later generalized by Yagasaki [11] to all values of $p > 2$. The approach in both of these works is to verify the Chicone Criterion which, while non-trivial, is not too difficult for the case when p is an integer. The problem is, however, surprisingly difficult when p is not an integer. Yagasaki first treats the case where p is rational and then extends the result to the general case by continuity. Yagasaki's treatment is a real tour de force and involves substantial amount of ingenuity and computations. A consequence of Yagasaki's result and also established in [11] is a complete bifurcation diagram for the system (1) with Neumann Boundary conditions $v'(\pm\pi) = 0$.

In view of the complexity of the arguments in [11] it is our aim to revisit this problem and give proofs that are, in our view, more elementary than the ones given in [11]. Once more we start with Chicone's criterion which amounts to check the convexity of a particular function. With simple changes of variables, this problem is then recast in terms of solutions of differential equations that can be understood via maximum principles.

2. The period function and Chicone's Criterion

By rescaling the solution of (1)

$$u(x) \to u\left(\sqrt{\frac{\lambda}{p-1}}x\right)$$

we may assume that u is a solution of the equation

$$u'' - u + u^p = 0 \tag{3}$$

with periodic boundary conditions on the interval $[0, T]$ where

$$T = 2\pi\sqrt{\frac{\lambda}{p-1}}. \tag{4}$$

Integrating this equation we get

$$\frac{u'^2}{2} + V(u) = E \tag{5}$$