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Ground-states for systems of M coupled semilinear Schrödinger equations with attraction–repulsion effects: Characterization and perturbation results

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ABSTRACT

We focus on the study of ground-states for the system of M coupled semilinear Schrödinger equations with power-type nonlinearities and couplings. We extend the characterization result in Correia (2016) to the case where both attraction and repulsion are present and cannot be studied separately. Furthermore, we derive some perturbation and classification results to study the general system where components may be out of phase. In particular, we present several conditions to the existence of nontrivial ground-states.

1. Introduction

In this work, we consider the system of M coupled semilinear Schrödinger equations

$$i(v_i)_t + \Delta v_i + \sum_{j=1}^{M} k_{ij}|v_j|^{p+1}|v_i|^{p-1}v_i = 0, \quad i = 1, \ldots, M \tag{1.1}$$

where $V = (v_1, \ldots, v_M) : \mathbb{R}^+ \times \Omega \to \mathbb{C}^M$, $\Omega \subset \mathbb{R}^N$ open with smooth boundary, $k_{ij} \in \mathbb{R}$, $k_{ij} = k_{ji}$, and $0 < p < 2/(N-2)^+$ (we use the convention $2/(N-2)^+ = +\infty$, if $N = 1, 2$, and $2/(N-2)^+ = 2/(N-2)$, if $N \geqslant 3$). Given $1 \leqslant i \neq j \leqslant M$, if $k_{ij} \geqslant 0$, one says that the coupling between the components v_i and v_j is attractive; if $k_{ij} < 0$, it is repulsive.

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When we look for nontrivial periodic solutions of the form $V = e^{i\omega t}U$, with $U = (u_1, \dots, u_M) \in (H_0^1(\Omega))^M$ (called bound-states), we are led to the study of the system

$$\Delta u_i - \omega u_i + \sum_{j=1}^{M} k_{ij}|u_j|^{p+1}|u_i|^{p-1}u_i = 0, \quad u_i \in H_0^1(\Omega),\ i = 1, \dots, M. \tag{M-NLS}$$

On the other hand, one may also consider periodic solutions where the time–frequency is not necessarily the same for each component (one then says that the components are out of phase). These solutions are of the form $V = (e^{i\omega_1 t}u_1, \dots, e^{i\omega_M t}u_M)$ and the stationary system is

$$\Delta u_i - \omega_i u_i + \sum_{j=1}^{M} k_{ij}|u_j|^{p+1}|u_i|^{p-1}u_i = 0, \quad u_i \in H_0^1(\Omega),\ i = 1, \dots, M. \tag{M-NLS$'$}$$

Notice that, if $M = 1$ and $\Omega = \mathbb{R}^N$, the presence of $\omega > 0$ may be eliminated by a suitable scaling. However, in any other case, such a procedure is no longer possible.

In any case, for both physical and mathematical reasons, one is interested in bound-states which have minimal action among all bound-states, the so-called ground-states. The set of such solutions is noted G. For $\Omega = \mathbb{R}^N$, in the scalar case, one may prove that there is a unique ground-state Q (modulo translations and rotations, see [1]). For a general Ω, the problem has not been completely solved. However, it is known, for example, that there exists a ground-state if

$$\omega > -\lambda_1(\Omega), \quad \lambda_1(\Omega) = \begin{cases} \text{first eigenvalue of } -\Delta \text{ on } H_0^1(\Omega), & \Omega \text{ bounded} \\ 0 & \Omega \text{ infinite parallelepiped.} \end{cases} \tag{1.2}$$

The vector case is much more complex. The existence of ground-states for system (M-NLS′) on $\Omega = \mathbb{R}^N$ has been proven under the sufficient and necessary condition

$$\exists U = (u_1, \dots, u_M) \in (H^1(\mathbb{R}^N))^M : \sum_{i,j=1} k_{ij} \int |u_i|^{p+1}|u_j|^{p+1} > 0 \tag{1.3}$$

using a suitable variational formulation. We note that the result is still true for any Ω bounded. In fact, one proves that the set of ground-states is the set of minimizers of

$$\inf \left\{ \int \sum_{i=1}^{M} \omega_i|u_i|^2 + |\nabla u_i|^2 : \sum_{i,j=1} k_{ij} \int |u_i|^{p+1}|u_j|^{p+1} = \lambda \right\}, \tag{1.4}$$

for a precise and explicit λ. To prove existence of minimizers, the main difficulty is the strong compactness of the minimizing sequence in L^{2p+2}. In Ω bounded, this is trivial, since one has the compact injection $H_0^1(\Omega) \hookrightarrow L^{2p+2}(\Omega)$. For $\Omega = \mathbb{R}^N$, one uses the concentration-compactness principle and proves the compactness alternative. Since the existence of ground-states for general Ω is an open problem, we shall make the following assumption

$$\text{“The set of all ground-states for (M-NLS) over } \Omega,\ G, \text{ is nonempty”.} \tag{Exist}$$

One then may pose a number of questions: is there a unique positive ground-state? Does a ground-state have all components different from 0 (called nontrivial ground-states)? Can we obtain a simple characterization of the family of ground-states? Are the solutions positive and radially decreasing?

Regarding system (M-NLS), for $\Omega = \mathbb{R}^N$, a recent work [2] has answered to these questions for a very large family of matrices $K = (k_{ij})_{1\leqslant i,j\leqslant M}$. Essentially, if one may group the components in such a way that two components attract each other if and only if they belong to the same group, one may answer all questions above in a satisfactory fashion. One may also prove that, in the case where all components attract each other, the result is extendible to any Ω. If this grouping hypothesis fails, the situation becomes much more difficult. The reason is that two components may repel each other directly but, by transitivity, they