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Nonlinear gradient estimates for generalized elliptic equations with nonstandard growth in nonsmooth domains

Sun-Sig Byun[a], Yumi Cho[b,*]

[a] Department of Mathematical Sciences and Research Institute of Mathematics, Seoul National University, Seoul 08826, Republic of Korea

[b] School of Mathematics, Korea Institute for Advanced Study, Seoul 02455, Republic of Korea

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ABSTRACT

A generalized elliptic equation with nonstandard growth in a nonsmooth bounded domain is studied. A global nonlinear gradient estimate is obtained in the frame of Orlicz spaces under the assumptions that the associated Young functions satisfy some moderate growth and decay conditions and that the boundary of the domain is δ-Reifenberg flat with δ depending on the domain, the dimension of the domain and the Young functions.

1. Introduction

In this paper we study a global estimate of Calderón–Zygmund type for the solution of a nonlinear elliptic problem with general growth and elliptic conditions in a bounded domain. The problem under consideration was introduced by Lieberman in [19] as a natural generalization of the p-Laplace equation with $1 < p < \infty$, and recently has been extensively investigated for the purpose of understanding finer regularity properties of solutions of a wider class of nonlinear equations of degenerate type, see [3,9,10,14,13,29,30] and the references therein.

As a typical example of nonlinear problems, the p-Laplace equation is the Euler–Lagrange equation of the energy functional

$$I[w] = \int_{\Omega} |Dw|^p \, dx,$$

* Corresponding author.
E-mail addresses: byun@snu.ac.kr (S.-S. Byun), yumicho@kias.re.kr (Y. Cho).

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where w is a function, Dw its gradient and Ω a domain in $\mathbb{R}^n$ with $n \geq 2$. It is natural to consider more general functional

$$I[w] = \int_\Omega G(|Dw|)\, dx$$

with a proper convex function G. If G is sufficiently smooth with $g = G'$, then one can derive its Euler–Lagrange equation

$$\operatorname{div}\left(\frac{g(|Dw|)}{|Dw|}Dw\right) = 0, \tag{1.1}$$

which is the so called generalized p-Laplace equation with G-growth. While the weak solutions of the p-Laplace equations are naturally in $W^{1,p}$ space, the weak solutions of (1.1) are in the class $W^{1,G}(\Omega)$ of functions $w : \Omega \to \mathbb{R}$ satisfying

$$\int_\Omega G(|Dw|)\, dx < \infty.$$

This generate type of the p-Laplace operator has been applied to various fields including Bifurcation problems, two phase free boundary problems, generalized Stokes system and Potential theory, as follows from [3,4,12,26]. There have been many results on regularity issues regarding elliptic and parabolic equations with the general G-growth type conditions. In [15] Esposito, Mingione and Trombetti showed that the solutions of elliptic equations with the nonlinearity vector fields $A(x, z)$ having strong monotone condition related N-function enjoy the Lipschitz regularity. They also dealt with the regularity problems for the minimizers of irregular integral functionals. Diening and Ettwein obtained fractional estimates for elliptic systems in [11]. In [13] Diening, Stroffolini and Verde discussed the $C^{1,\alpha}$-regularity for the functionals. Cianchi and Maz'ya proved the global Lipschitz regularity for the elliptic equations and systems in [9,10]. Very recently Baroni [3] considered nonlinear elliptic equations with measure data to prove gradient bounds for solutions in terms of linear Riesz potentials.

In this paper we are mainly interested in establishing a global Calderón–Zygmund theory for the generalized p-Laplace equation with G-growth and a datum of divergence type on the right hand side over a bounded domain having a very rough boundary. In [29] Verde proved a Calderón–Zygmund theory in the whole domain $\mathbb{R}^n$ for the elliptic system of the general type by showing that if the datum belongs to any Orlicz space with some proper conditions on the associated Young function, then the gradient of a solution also belongs to the same Orlicz space. A main analytic tool used in [29] is the sharp maximal function of Fefferman–Stein. Motivated by this work we study a global gradient integrability estimate to the generalized p-Laplace elliptic problem on a very irregular bounded domain.

The Calderón–Zygmund theory as a fundamental and classical topic in the field of partial differential equations is to study how the regularity of solutions is influenced by the one of given data on the right hand side and to derive sharp estimates under possibly minimal regularity assumptions on the relevant differential operators such as the coefficients and the underlying domains, see [21,22]. Indeed, there have been many notable results in the literature. We here mention some nonlinear results in the literature. In [16] the interior L^q estimate with $p \leq q < \infty$ for the gradient of a solution to the p-Laplace elliptic equation was obtained. For the parabolic case, E. Acerbi and G. Mingione showed Calderón–Zygmund type estimates in [1]. The paper [7,23] treated the nonlinear elliptic problems of the general type with p-growth for the interior L^q estimate and a global L^q estimate on the whole domain $\mathbb{R}^n$, respectively. In [5] the global weighted L^q estimate for the general form of p-Laplace elliptic equation was obtained with the Muckenhoupt weight and over a nonsmooth bounded domain.

As mentioned earlier, our purpose is to prove that the gradient of solutions has the same integrable regularity as that of the given data which belong to any Orlicz space with the Young function satisfying