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Regularity and structure of pullback attractors for reaction–diffusion type systems without uniqueness

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ABSTRACT

In this paper, we study the pullback attractor for a general reaction–diffusion system for which the uniqueness of solutions is not assumed. We first establish some general results for a multi-valued dynamical system to have a bi-spatial pullback attractor, and then we find that the attractor can be backwards compact and composed of all the backwards bounded complete trajectories. As an application, a general reaction–diffusion system is proved to have an invariant (H, V)-pullback attractor $A = \{A(\tau)\}_{\tau\in\mathbb{R}}$. This attractor is composed of all the backwards compact complete trajectories of the system, pullback attracts bounded subsets of H in the topology of V, and moreover

$$\bigcup_{s\leqslant\tau} A(s) \quad \text{is precompact in } V, \quad \forall \tau \in \mathbb{R}.$$

A non-autonomous Fitz-Hugh–Nagumo equation is studied as a specific example of the reaction–diffusion system.

1. Introduction

A multi-valued non-autonomous dynamical system (m-NDS) is generated by an evolution equation whose solution is unnecessarily unique. The study on m-NDS has particular significance since m-NDS are often established under weaker conditions than usual (single-valued) dynamical systems. For an m-NDS ϕ on a Banach space X, a pullback attractor $A = \{A(\tau)\}_{\tau\in\mathbb{R}}$ is defined as a compact and negatively invariant non-autonomous set which attracts each bounded subset of X (see Definition 2.7). This can be regarded as a generalization of pullback attractors for single-valued dynamical systems which have been studied widely, see for instance [3–5,11].

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In this paper, we study the structure and regularity of the pullback attractor for the m-NDS generated by the following reaction–diffusion equation defined on a bounded domain $\mathcal{O} \subset \mathbb{R}^N$:

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} - a\triangle u + f(t,u) = g(x,t), \\ u|_{\partial\mathcal{O}} = 0, \\ u|_{t=\tau} = u_0, \end{cases} \tag{1.1}$$

where $(t,x) \in (\tau,\infty) \times \mathcal{O}$, $u = (u^1(t,x),\ldots,u^d(t,x)) : (\tau,\infty) \times \mathcal{O} \to \mathbb{R}^d$, a is a real $d \times d$ matrix with a positive symmetric part $\frac{a+a^t}{2} \geqslant \beta I$ for some $\beta > 0$, and $f(t,u) \in C((\tau,\infty)\times\mathbb{R}^d;\mathbb{R}^d)$ satisfies some conditions unable to ensure the uniqueness of solutions (see Section 4). The most direct example of this system is the classical reaction–diffusion equation (taking $d = 1$) frequently seen in the literature. Nevertheless, the system (1.1) covers more models, such as the Fitz-Hugh–Nagumo equations considered in Section 4.4, and others stated by [16,15].

We get three aims in this work. The first is the existence of the (H,V)-pullback attractor $A = \{A(\tau)\}_{\tau\in\mathbb{R}}$ for (1.1), which is a non-autonomous set pullback attracting bounded sets of H under the topology of V, where

$$H := (L^2(\mathcal{O}))^d, \qquad V := (H_0^1(\mathcal{O}))^d.$$

This is a study of bi-spatial attractors which attracted much attention these years due to their higher regularity and stronger attracting ability compared with usual attractor, see [7,13,12] for single-valued non-autonomous/random cocycles and [21,19] for multi-valued semi-groups and random cocycles, respectively. In this paper, we develop a study for m-NDS which can be regarded as an extension of the bi-spacial attractor theory on one hand, and is interesting because of the multi-valued feature on the other hand.

The second aim is to establish the backwards precompactness of A, that is,

$$\bigcup_{s\leqslant\tau} A(s) \text{ is precompact in } V, \quad \forall\tau\in\mathbb{R}.$$

This subject is new since the compactness of a pullback attractor is often considered for each fixed "section" $A(\tau)$ in the literature. To achieve this, we first establish some sufficient conditions to ensure an attractor to be backwards precompact, and then apply them to the system (1.1) when the external force g is backwards translation bounded, namely, $g \in L^2_{loc}(\mathbb{R};H)$ with

$$\sup_{\tau<0}\int_{\tau-1}^{\tau} \|g(s)\|^2 \,\mathrm{d}s < \infty.$$

This condition is shown weaker than translation bounded condition considered in [8] and stronger than tempered conditions seen in recent studies for random dynamical systems, such as [17] and references therein. Establishing a non-autonomous set $\{K(\tau)\}_{\tau\in\mathbb{R}}$ which is absorbing and increasing in τ plays a key role to obtain the backwards compactness of the attractor.

The third main aim is to characterize the pullback attractor A by complete trajectories. For strict m-NDS we first introduce the concept of an m-NDS generating smooth trajectories, and then prove that backwards bounded pullback attractors for such systems are composed of backwards bounded trajectories, see Theorem 2.24. Note that, the class of dynamical systems who generate smooth trajectories is rather general. A direct example is the so-called *generalized dynamical systems*, first studied by Ball [1] and recently studied by Simsen [14] for generalized semiflows and Kapustyan et al. [9] for generalized processes, etc. Therefore, Theorem 2.24 is interesting even in single-valued cases as single-valued systems must be strict.

We carry out this work as follows.