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Blow up boundary solutions of some semilinear fractional equations in the unit ball

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A B S T R A C T

For $\gamma > 0$, we are interested in blow up solutions $u \in \mathscr{C}^+(B)$ of the fractional problem in the unit ball B. We distinguish particularly two orders of singularity at the boundary: solutions exploding at the same rate than $\delta^{\frac{\alpha}{2}-1}$ (δ denotes the Euclidean distance) and those higher singular than $\delta^{\frac{\alpha}{2}-1}$. As a consequence, it will be shown that the classical Keller–Osserman condition cannot be readopted in the fractional setting.

1. Introduction

Let $N \geq 3,\ 0 < \alpha < 2$ and let $(X_t)_{t\geq 0}$ be the standard α-stable process in $\mathbb{R}^N$. It is determined by its characteristic function which takes the form

$$E^x\left(e^{i\xi(X_t - X_0)}\right) = e^{-t|\xi|^\alpha}; \quad \xi \in \mathbb{R}^N,$$

where E^x is the expectation with respect to the distribution P^x of the process starting from $x \in \mathbb{R}^N$. It is a discontinuous Markov process and gives rise to equations with the fractional Laplacian $\Delta^{\frac{\alpha}{2}}$.

Nonlocal operators such as $\Delta^{\frac{\alpha}{2}}$ naturally arise in population dynamics, continuum mechanics, game theory and some other fields, we quote for instance [19,28,30,34,35], one can see also [10] for broader discussion.

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In the classical setting, i.e. for $\alpha = 2$, one of the most commonly considered equations in the literature is the following:

$$\Delta u = \varphi(u) \quad \text{in } B, \tag{1}$$

where B is the unit ball of $\mathbb{R}^N$ and $\varphi : [0, \infty[\to \mathbb{R}$ is some nonnegative nondecreasing function. Such equations appear naturally in several interesting contexts including in the random systems of branching particles [18,27]. Solutions of (1) verifying

$$\lim_{x \to \partial B} u(x) = \infty$$

are called large solutions or blow up (boundary) solutions. Merely for the sake of completeness, we shall recall the pioneering work of [24,31]. They proved independently that a necessary and sufficient condition for the existence of a large solution to (1), where φ is a positive nondecreasing function, is

$$\int_1^\infty \left(\int_0^s \varphi(t)\, dt \right)^{-\frac{1}{2}} ds < \infty,$$

that in the case of $\varphi(u) = u^\gamma$, means $\gamma > 1$.

The aim of the paper is to study (1) substituting the classical Laplacian by one of its fractional powers. Our consideration is motivated by the natural question whether the classical results (particularly the Keller–Osserman condition) in this field may be extended to nonlocal operators.

The behavior of α-harmonic functions contrasts, in some respects, with the one of the classical harmonic functions. Indeed, due to the jumping nature of the α-stable process, roughly speaking, at the exit time one could end up anywhere outside the domain. Put differently, the process typically leaves domains by jumping to the interior of its complement while the continuous paths of Brownian motion leave domains by hitting the boundary. This spans the existence of positive harmonic functions on B blowing up at the boundary. Such functions are called singular α-harmonic functions and they are harmonic for the stable process $(X_t^B)_{t \geq 0}$ killed on leaving B. In the Brownian motion case, such functions do not exist due to the Fatou theorem and nontangential convergence of positive harmonic functions [2,17,36]. In this sense, the Martin kernel M_B^α

$$\nu \to \int_{\partial B} \frac{(1 - |x|^2)^{\frac{\alpha}{2}}}{|x - y|^N} \nu(dy); \quad x \in B$$

provides a one-to-one correspondence between positive Random measures ν on ∂B and positive α-harmonic functions on B which supports the fact that singular α-harmonic functions constitute, in some respects, the appropriate class of "harmonic functions". Nevertheless, the usual probabilistic interpretation of singular α-harmonic functions on B as solutions of Dirichlet problem is no longer true. The interested reader is referred to [14, Theorem 3.18] where the authors provide some probabilistic interpretation of these functions. In particular,

$$M_B^\alpha 1(x) := \int_{\partial B} \frac{(1 - |x|^2)^{\frac{\alpha}{2}}}{|x - y|^N} \sigma(dy); \quad x \in B,$$

defines a singular α-harmonic on B and it behaves like $\delta(x)^{\frac{\alpha}{2}-1}$ on B. Here, σ denotes the surface area measure on ∂B and $\delta(x) := 1 - |x|$ is the Euclidean distance from x to the boundary ∂B. This entitles us to study the following appropriate reformulated semilinear Dirichlet problem associated to $\Delta^{\frac{\alpha}{2}}$ in B taking into account the aspects raised above.

$$\begin{cases} \Delta^{\frac{\alpha}{2}} u = u^\gamma \quad \text{in } B \\ u = 0 \quad \text{in } B^c \\ \lim\limits_{x \to z} \delta(x)^{1-\frac{\alpha}{2}} u(x) = g(z) \end{cases} \tag{2}$$