Nonlinear Analysis 130 (2016) 1–17

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Some geometrical properties of convex level sets of minimal graph on 2-dimensional Riemannian manifolds

Peihe Wang*, Lingling Zhao

School of Math. Sci., Qufu Normal University, 273165, Qufu Shandong, China

ARTICLE INFO

Article history:
Received 10 May 2015
Accepted 21 September 2015
Communicated by S. Carl

MSC:
35J05
53J67

Keywords:
Level sets
Curvature estimate
Minimal graph

ABSTRACT

For the minimal graph defined on two dimensional Riemannian manifolds with constant Gauss curvature, we derive a constant rank theorem on the geodesic curvature of its level sets, and an auxiliary function involving the curvature of the level sets will be found to obtain some differential equalities to study the geometrical properties.

1. Introduction

As is known to us, there are lots of interesting results about the geometric properties of the level sets of the solutions to elliptic partial differential equations. For instance, Korevaar [5], Xu [15] and Bian–Guan–Ma–Xu [1] derived the constant rank theorems of the level sets of the solution to some partial differential equations with suitable structure conditions. On the other hand, Ortel–Schneider [11], Longinetti [6,7] proved that the curvature of the level curves attains its minimum on the boundary (see also Talenti [12] for related results) for 2-dimensional harmonic function with convex level curves. Furthermore, Longinetti studied the precise relation between the curvature of the convex level lines and the height of minimal graph in [7]. The curvature estimates of the level sets of the solution to partial differential equations then have no new progress until recently, Ma–Ou–Zhang [8] got the Gaussian curvature estimates of the convex level sets of harmonic functions which depend on the Gaussian curvature of the boundary and the norm of the gradient on the boundary in R^n. Furthermore, in [9] the concavity of the Gaussian curvature of the convex level sets of p-harmonic functions with respect to the height was derived to describe the variation

* Corresponding author.
E-mail address: peihewang@hotmail.com (P. Wang).

http://dx.doi.org/10.1016/j.na.2015.09.021

of the curvature along the height of the function. In [4], the lower bound of the principle curvature of the convex level sets of the solution to a kind of fully nonlinear elliptic equations was derived. For Poisson equations and a class of semilinear elliptic partial differential equations, Caffarelli–Spruck [2] concluded that the level sets of their solutions are all convex with respect to the gradient direction, the curvature estimates of the level sets have been got by Wang–Zhang [14], and in the same paper they also described the geometrical properties of the level sets of the minimal graph. In the sequel, following the technique in [9], Wang [13] got the precise relation between the curvature of the convex level sets and the height of minimal graph of general dimension which generalized the previous results of Longinetti [7].

In their original paper, Wang–Zhang [14] got the following theorem to describe the sharp curvature estimate of the level sets of minimal graph.

Theorem 1.1 ([14]). *Let Ω be a smooth bounded domain in R^2. Let $u \in C^4(\Omega)\bigcap C^2(\overline{\Omega})$ be the solution to the following minimal surface equation,*

$$\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)=0, \quad \text{in } \Omega \subset \mathrm{R}^2.$$

Assume $|\nabla u| \neq 0$ in Ω. If the level sets of u are strictly convex with respect to the normal direction ∇u, and let K be the curvature of the level sets, then $(\frac{|\nabla u|^2}{1+|\nabla u|^2})^{-\frac{1}{2}}K$ attains its minimum and maximum on the boundary $\partial\Omega$.

For the important Riemannian manifolds cases, the geometric properties of the level sets of the harmonic function defined on space forms, Ma–Zhang [10] derived the strict convexity and the lower bound of the Gaussian curvature. This is an interesting result about the harmonic functions on space forms.

Combining the results above, it is a natural question to generalize this problem to the minimal graph defined on space forms. In this paper, we come firstly to consider the two dimensional case and derive an analogue of the results in [10] and in [14]. We will prove the following strictly convexity theorem and derive the curvature estimate of the level sets:

Theorem 1.2. *Let Ω be a smooth bounded connected domain on the two dimensional space form M^2 with constant curvature ϵ. Let $u \in C^4(\Omega)\bigcap C^2(\overline{\Omega})$ be the minimal graph defined on Ω, i.e. $\mathrm{div}(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}})=0$. Assume $|\nabla u| \neq 0$ in Ω and the level sets of u are convex with respect to the normal ∇u, then they are strictly convex.*

Theorem 1.3. *Let Ω be a smooth bounded domain on the two dimensional manifolds M^2 with constant curvature ϵ. Let $u \in C^4(\Omega)\bigcap C^2(\overline{\Omega})$ be the minimal graph defined on Ω, i.e. $\mathrm{div}(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}})=0$. Assume $|\nabla u| \neq 0$ and the level sets of u are convex with respect to ∇u. Let k_g be the geodesic curvature of the level sets of u in Ω. Then we have*

(i) *For $\epsilon > 0$, the function $(\frac{|\nabla u|^2}{1+|\nabla u|^2})^{-\frac{1}{2}}k_g$ attains its minimum on the boundary $\partial\Omega$;*
(ii) *For $\epsilon = 0$, the function $(\frac{|\nabla u|^2}{1+|\nabla u|^2})^{-\frac{1}{2}}k_g$ attains its maximum and minimum on the boundary $\partial\Omega$;*
(iii) *For $\epsilon < 0$, the function $(\frac{|\nabla u|^2}{1+|\nabla u|^2})^{-\frac{1}{2}}k_g$ attains its maximum on the boundary $\partial\Omega$.*

Remark that the conclusion (ii) coincides with the theorem in [14]. Also remark that these results give us a chance to understand the geometrical properties of the level sets of the minimal surface defined on space forms.

The paper is organized as follows: in Section 2, we list the notations and the preliminaries being used during the process of the proof, especially, we deduce the equation of the minimal graph on Riemannian