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Evolution of convex hypersurfaces by a fully nonlinear flow

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ABSTRACT

In this paper, we study the evolution of convex hypersurfaces by a fully nonlinear function of curvature minus an external force field c. We prove that the flow will preserve the convexity for any c. When $c < g$ on the initial surface, where g is the fully nonlinear function, we prove that the flow will expand the hypersurface for all time. If on initial surface M_0 the minimal principal radius of curvature $r_{\min}$ satisfies $r_{\min} > nc$ and minimum of the support function s satisfies $s_{\min} > nc$, then after a scaling the hypersurface will converge to a sphere. If $c > g$ on the initial surface M_0 and the diameter of M_0 satisfies $\operatorname{diam}(M_0) < \sqrt{2}nc$, we show that the maximal existence time of the flow is finite.

1. Introduction

Let M_0 be a smooth, closed and strictly convex hypersurface in an Euclidean space $\mathbb{R}^{n+1}$. Suppose that M_0 is given by a smooth embedding $F_0 : S^n \to \mathbb{R}^{n+1}$. In this paper, we study the flow

$$\frac{dF}{dt} = (k(x,t) - c)\nu := f\nu, \tag{1.1}$$

where $k(\cdot,t)$ is a suitable curvature function of hypersurface M_t parametrized by $F(\cdot,t) : S^n \to \mathbb{R}^{n+1}$. ν is the outer unit normal vector, and c is a constant.

We assume that the curvature function $k(\cdot,t)$ can be expressed as

$$k(\cdot,t) = g(r_1,\ldots,r_n) \tag{1.2}$$

where $r_1,\ldots,r_n$ are the principal radii of curvature of the hypersurface M_t, and $g \in C^\infty(\Gamma^+)$ is a positive, symmetric function on the positive cone $\Gamma^+ = \{(r_1,\ldots,r_n) \in \mathbb{R}^n : r_i > 0 \text{ for all } i\}$. The function g is assumed to satisfy the following conditions:

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(A1) g is homogeneous of degree one on Γ^+,
(A2) $\frac{\partial g}{\partial r_i} > 0$ on Γ^+,
(A3) g is concave on Γ^+.

We also assume that $g(1,\ldots,1) = \frac{1}{n}$. The reason we choose $\frac{1}{n}$ as the normalized constant is if g is the inverse mean curvature function, then $g(1,\ldots,1) = \frac{1}{n}$. If we let $g(1,\ldots,1) = \beta$ for some positive constant β, the main conclusions of this paper still hold.

For $c = 0$, flow (1.1) has been studied in [3,15,16]. In [16], Urbas proved that the surfaces stay strictly convex and smooth for all time under the flow. Furthermore, the surfaces become more and more spherical in the process. In [3,15], the authors got the similar results for the star-shaped initial hypersurface. If g is the inverse mean curvature function, flow (1.1) is called inverse mean curvature flow with forced term, which has been studied in [11]. In [11], the author proved that the convexity of the flow (1.1) is preserved and the hypersurface contracts if $c > g$ and expands if $c < g$. For the study of inverse mean curvature flow, one can refer to [5,6].

The curvature flows with forced term have been studied extensively. For mean curvature flow with external force field, which is a model describing the Ginzburg–Landau vortex, one can refer to [7,9,10,12–14]. There are also some other evolution equations for geometric quantities for general speeds, see e.g. [2,4].

In this paper, we will study the properties of the hypersurfaces under flow (1.1), we also want to find out how the force term c affects the flow. First, we prove that when $c > g$ on M_0 and the diameter M_0 satisfying $\mathrm{diam}(M_0) < \sqrt{2}nc$, flow (1.1) will exist in finite time.

Theorem 1.1. *Let M_0 be a smooth, closed and strictly convex hypersurface in $\mathbb{R}^{n+1}$, $n > 2$, given by a smooth embedding $F_0 : S^n \to \mathbb{R}^{n+1}$. Let $g \in C^\infty(\Gamma^+)$ be a positive, symmetric function on the positive cone satisfying* (A1)–(A3). *If $c > g$ on M_0 and the diameter of M_0 satisfies diam$(M_0) < \sqrt{2}$nc, then the flow* (1.1) *will exist in finite time.*

When $c < g$ on M_0 and g satisfies an extra condition, we will prove flow (1.1) will expand for all time. Let $r_{\min}$ denote the minimal principal curvature radius of the convex hypersurface, and $s_{\min}$ denote the minimum of support function, then if $r_{\min} > nc$ and $s_{\min} > nc$ on M_0, we obtain that the surface will converge to sphere after a scaling.

Theorem 1.2. *Let M_0 be a smooth, closed and strictly convex hypersurface in $\mathbb{R}^{n+1}, n > 2$ given by a smooth embedding $F_0 : S^n \to \mathbb{R}^{n+1}$. Let $g \in C^\infty(\Gamma^+)$ be a positive, symmetric function on the positive cone satisfying* (A1)–(A3). *If $c < g$ on M_0, and one of the following conditions is satisfied*

(B1) *$g \in C^0(\overline{\Gamma^+})$ and $g \equiv 0$ on $\partial\Gamma^+$;*
(B2) *the function g defined by*

$$g(r_1,\ldots,r_n) = \frac{1}{w\left(\frac{1}{r_1},\ldots,\frac{1}{r_n}\right)}$$

is concave on Γ^+.

Then the flow (1.1) *will expand the convex hypersurface for all time. And if $r_{\min} > nc$ and $s_{\min} > nc$ on M_0, and let $\widetilde{M_t} = e^{-\frac{1}{n}t}M_t$, then $\widetilde{M_t}$ will converge in C^∞ topology to a sphere.*

When $g = \frac{1}{H}$, where H is the mean curvature function, the same conclusions as Theorems 1.1 and 1.2 have been proved by the author in [11]. We want to point out here that in that paper, when proving the convergence of the scaling surfaces, we missed the condition $r_{\min} > nc$ and $s_{\min} > nc$ on M_0.