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The density of solutions to multifractional stochastic Volterra integro-differential equations

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ABSTRACT

The aim of this paper is to study the density of solutions to stochastic Volterra integro-differential equations with multifractional noises. We prove the existence of the density of solutions and show that the density is bounded by lower and upper Gaussian bounds.

1. Introduction

It is well known that fractional Brownian motion (fBm) has many applications in various fields. For example, it is effectively used as a driving noise in finance (see, for instance [6]), in biology (see, [5]), in communication networks (see, [23]), etc. One of the most important generalizations of fBm is the multifractional Brownian motion (mBm), which was first introduced by Peltier and Véhel in [21] and by Benassi et al. in [2]. This class of stochastic processes owns more flexible properties than that of fBm and hence, it overcomes the existing limitations in the models with fBm (see, e.g. [3,13]).

Because of its applications, one will want to develop systematically the stochastic calculus with respect to mBm. In fact, the rigorous definitions of stochastic integrals with respect to mBm have been recently provided in [11,12]. The existence and uniqueness of solutions to some specific stochastic differential equations can be found in [8,9]. However, there are still a lot of different aspects that need studying. Motivated by this observation, the aim of the present paper is to develop the work started in [8].

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There are several definitions of multifractional Brownian motion. One of them is a so-called multifractional Brownian motion of the Riemann–Liouville form. This process was introduced first by Lim and Muniandy in [15], its definition is as follows.

Definition 1.1. Let $H : [0, +\infty) \to (0, 1)$ be a Hölder function of exponent $\beta > 0$. A multifractional Brownian motion of the Riemann–Liouville form (RL-mBm) with Hurst functional parameter $H \in (0, 1)$, denoted by $\{W_t^{H_t}, t \geq 0\}$, is a centred Gaussian process defined by

$$W_t^{H_t} = \int_0^t K(t,s) dW_s, \tag{1.1}$$

where W is a standard Brownian motion and $K(t,s) = (t-s)^{\alpha_t},\ \alpha_t = H_t - \frac{1}{2}$ is a Volterra kernel, i.e. $K(t,s) = 0$ for all $s \geq t$.

We note that RL-mBm is a generalization of fBm of the Riemann–Liouville form which has an excellent application in finance (see, e.g. [7]). On the other hand, Lim [14] pointed out that RL-mBm shares many properties of mBm defined by Peltier and Véhel and that RL-MBM can be a better candidate than mBm in modelling and simulation because of its simple representation. We therefore choose to use RL-mBm standing for mBm in our studies.

In the recent paper [8], we based on the method developed by Alòs et al. in [1] to introduce a definition of stochastic integrals with respect to RL-mBm. Then, we study linear multifractional stochastic Volterra integro-differential equations that are described in form

$$dX_t = \left(aX_t + \int_0^t G(t-u) X_u du \right) dt + \sigma_t dW_t^{H_t}, \quad t \in [0, T], \tag{1.2}$$

the initial condition $X_0 = x$ is a real constant, where $(W_t^{H_t})_{t \in [0,T]}$ is a RL-mBm with $H_t \in (\frac{1}{2}, 1)$ and is a continuously differentiable function on $[0, T]$.

For Eq. (1.2), its solution is unique and admits the following representation which is the so-called the variation of parameters formula (see, Theorem 3.1 in [8])

$$X_t = Z_t X_0 + \int_0^t Z_{t-s} \sigma_s dW_s^{H_s}, \quad t \in [0, T], \tag{1.3}$$

where Z_t is the unique solution of the resolvent equation

$$dZ_t = \left(aZ_t + \int_0^t G(t-u) Z_u du \right) dt, \quad Z_0 = 1.$$

It follows from (1.3) that if $\{\sigma_s, s \in [0, T]\}$ is a deterministic function, then the solution X_t of the linear integro-differential equations (1.2) is a Gaussian random variable for each $t \in (0, T]$. Let us now consider non-linear stochastic integro-differential equations of the form

$$dX_t = \left(a(t, X_t) + \int_0^t G(t-u) b(u, X_u) du \right) dt + \sigma_t dW_t^{H_t}, \quad t \in [0, T], \tag{1.4}$$

the initial condition $X_0 = x$ is a real constant.

In general, the law of the solution $(X_t)_{t \in [0,T]}$ to (1.4) is not Gaussian even when $G \equiv 0$ and RL-mBm $(W_t^{H_t})_{t \in [0,T]}$ reduces to a standard Brownian motion (i.e. $H_t \equiv \frac{1}{2}$). Because the density function is one of the most natural features for any random variable, it would be desirable to study the density of solutions to (1.4). In fact, this is a well known topic in the theory of stochastic differential equations. We refer the reader to [17] for an extensive survey of the existing literature and to the paper [4,18,19] for the results about Gaussian density estimates. However, to the best of our knowledge, the study of this topic for the class of multifractional stochastic integro-differential equations has not yet been addressed.