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# The Cauchy problem for the Hartree type equation in modulation spaces

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ABSTRACT

We study the Cauchy problem for Hartree equation with cubic convolution nonlinearity $F(u) = (K * |u|^2)u$ under a specified condition on potential $K$ with Cauchy data in modulation spaces $M^{p,q}(\mathbb{R}^d)$. We establish global well-posedness results in $M^{1,1}(\mathbb{R}^d)$ when $K(x) = \lambda|x|^{-\gamma}(\lambda \in \mathbb{R}, 0 < \gamma < \min\{2, d/2\})$; in $M^{p,q}(\mathbb{R}^d)$ $(1 \leq q \leq \min\{p, p'\}$ where $p'$ is the Hölder conjugate of $p \in [1,2])$ when $K$ is in Fourier algebra $\mathcal{F}L^1(\mathbb{R}^d)$, and local well-posedness result in $M^{p,1}(\mathbb{R}^d)$ $(1 \leq p \leq \infty)$ when $K \in M^{1,\infty}(\mathbb{R}^d)$.



## 1. Introduction

Inspiring from the work of Chadam–Glassey [7] in 1980s Ginibre–Velo [11] have studied the Shrödinger equation with cubic convolution nonlinearity due to both their strong physical background and theoretical importance (for instance, it appears in quantum theory of boson stars, atomic and nuclear physics, describing superfluids, etc.). This model is known as the Hartree type equation:

$$iu_t + \Delta u = (K * |u|^2)u, \quad u(x, t_0) = u_0(x); \tag{1.1}$$

where $u(x,t)$ is a complex valued function on $\mathbb{R}^d \times \mathbb{R}$, $\Delta$ is the Laplacian in $\mathbb{R}^d$, $u_0$ is a complex valued function on $\mathbb{R}^d$, $K$ is some suitable potential (function) on $\mathbb{R}^d$, time $t_0 \in \mathbb{R}$, and $*$ denotes the convolution in $\mathbb{R}^d$.

In subsequent years the local and global well-posedness, regularity, and scattering theory for Eq. (1.1) have attracted a lot of attention by many mathematicians. Almost exclusively, the techniques developed so far restrict to Cauchy problems with initial data in a Sobolev space, mainly because of the crucial role played by the Fourier transform in the analysis of partial differential operators. See [5,11,6].

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We note that over the past ten years there has been increasing interest for many mathematicians to consider Cauchy data in modulation spaces $M^{p,q}(\mathbb{R}^d)$ (Definition 2.1) for nonlinear dispersive equations because these spaces are rougher than any given one in a fractional Bessel potential space and this low-regularity is desirable in many situations. For instance, we mention, the local well-posedness result of Shrödinger equation, especially, with power type nonlinearity $F(u) = |u|^{2k}u$ ($k \in \mathbb{N}$) are obtained in [18,3,8,4] with Cauchy data from $M^{p,1}(\mathbb{R}^d)$ and a global existence result in [17,13] with small initial data from $M^{p,1}(\mathbb{R}^d)$ $(1 \leq p \leq 2)$. See also [9,14]. However, the global well-posedness result for the large initial data (without any restriction to initial data) in modulation space is not yet clear, in fact it is an open question [15, p. 280], because one of the main obstacle is a lack of useful conservation laws in modulation spaces by which one can guarantee the global existence result.

Taking these considerations into our account, in this article, we will investigate Hartree type equation (1.1) with potentials are of the following type:

$$K(x) = \frac{\lambda}{|x|^{\gamma}}, \quad (\lambda \in \mathbb{R}, \gamma > 0,\ x \in \mathbb{R}^d), \tag{1.2}$$

$$K \in \mathcal{F}L^1(\mathbb{R}^d), \tag{1.3}$$

$$K \in M^{1,\infty}(\mathbb{R}^d). \tag{1.4}$$

The homogeneous kernel of the form (1.2) is known as Hartree potential. Now we note that the solutions to (1.1) enjoy (for instance see Proposition 2.8) the mass conservation law,

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2} \quad (t \in \mathbb{R}),$$

and exploiting this mass conservation law and techniques from time–frequency analysis we prove global existence result (Theorem 1.1) for Eq. (1.1) in the space $M^{1,1}(\mathbb{R}^d)$ for $K$ of the form (1.2); the proof relies on some suitable decomposition of Fourier transform of Hartree potential into Lebesgue spaces (Eq. (3.1)). We prove global existence result (Theorem 1.2) in the space $M^{p,q}(\mathbb{R}^d)$ when potential $K \in \mathcal{F}L^1(\mathbb{R}^d)$ (definition (2.6)) and local existence (Theorem 1.3) via uniform estimates for the Shrödinger propagator in modulation spaces $M^{p,q}(\mathbb{R}^d)$ and algebraic properties of the space $M^{p,q}(\mathbb{R}^d)$.

We state our main results:

**Theorem 1.1.** *Assume that $u_0 \in M^{1,1}(\mathbb{R}^d)$ and let $K$ be given by (1.2) with $\lambda \in \mathbb{R}$, and $0 < \gamma < \min\{2, d/2\}$, $d \in \mathbb{N}$. Then there exists a unique global solution of (1.1) such that $u \in C(\mathbb{R}, M^{1,1}(\mathbb{R}^d))$.*

**Theorem 1.2.** *Let $K \in \mathcal{F}L^1(\mathbb{R}^d), d \in \mathbb{N}$. Then, for any $u_0 \in M^{p,q}(\mathbb{R}^d)$ $(1 \leq q \leq \min\{p, p'\}$ where $p'$ is the Hölder conjugate of $p \in [1, 2])$, there exists a unique global solution $u(t)$ of (1.1) such that $u(t) \in C(\mathbb{R}, M^{p,q}(\mathbb{R}^d))$.*

**Theorem 1.3.** *Assume that $u_0 \in M^{p,1}(\mathbb{R}^d)$ $(1 \leq p \leq \infty)$, and $K \in M^{1,\infty}(\mathbb{R}^d), d \in \mathbb{N}$. Then, there exist $T^* = T^*(\|u_0\|_{M^{p,1}}) > t_0$ and $T_* = T_*(\|u_0\|_{M^{p,1}}) < t_0$ such that (1.1) has a unique solution $u \in C([T_*, T^*], M^{p,1}(\mathbb{R}^d))$.*

**Remark.** The analogue of Theorem 1.3 holds for the general nonlinearity $(K * |u|^{2k})u, k \in \mathbb{N}$, that is, for the Shrödinger equation with the nonlinearity $(K * |u|^{2k})u, k \in \mathbb{N}$.

## 2. Preliminaries and notations

The notation $A \lesssim B$ means $A \leq cB$ for a some constant $c > 0$, whereas $A \asymp B$ means $c^{-1}A \leq B \leq cA$, for some $c \geq 1$. The symbol $A_1 \hookrightarrow A_2$ denotes the continuous embedding of the topological linear space $A_1$