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Long time existence for the quadratic wave equation associated to the harmonic oscillator☆

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ABSTRACT

We show the solution for the quadratic wave equation associated to the harmonic oscillator exists over a longer time interval than the one given by local existence theory. The key point is that the nonlinearity is quadratic so that we can estimate the small divisor and perform a normal form process.

1. Introduction

The paper is devoted to get lower bounds for the lifespan of the solution to the quadratic wave equation associated to the harmonic oscillator:

$$\begin{cases} \partial_t^2 u - \Delta u + |x|^2 u = u^2, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d \\ u(t,x)|_{t=0} = \epsilon f(x), & \partial_t u(t,x)|_{t=0} = \epsilon g(x) \end{cases} \tag{1.1}$$

where integer d is the space dimension, $\epsilon > 0$ is small and $(f,g) \in \mathscr{H}^s(\mathbb{R}^d) \times \mathscr{H}^{s-1}(\mathbb{R}^d)$. Here $\mathscr{H}^s(\mathbb{R}^d)$ is defined for natural number s by

$$\mathscr{H}^s(\mathbb{R}^d) = \{u \in L^2(\mathbb{R}^d) : x^\alpha \partial^\beta u \in L^2(\mathbb{R}^d),\ \forall \alpha, \beta \in \mathbb{N}^d \text{ satisfying } |\alpha| + |\beta| \le s\}. \tag{1.2}$$

Let us recall some results about the similar equation

$$\partial_t^2 u - \Delta u = u^2 \tag{1.3}$$

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with compactly supported small smooth data. It has global solution when $d > 4$ (see [1,9]). The solution blows up in dimension $d \le 4$ (see [2,4,10,14]). These results are covered by the celebrated Strauss conjecture. When blow-up happens, one also interests in getting upper and lower bounds of the lifespan $T(\epsilon)$ of solutions. Almost global existence result has been obtained in dimension $d = 4$ in [7,8,15]. While in low dimensions $d = 3, 2$ and some particular case of one dimension, it was shown in [6,13,12]

$$T(\epsilon) \simeq \epsilon^{-\frac{2}{4-d}}. \tag{1.4}$$

The situation is drastically different when we replace $-\Delta$ by $-\Delta + |x|^2$ since the latter operator has pure point spectrum. Actually, the spectrum of $-\Delta + |x|^2$ is included in $2\mathbb{N} + 1$. This prevents any time decay for solutions of the linear equation, which is essential in obtaining estimates of the lifespan. Moreover, due to the appearance of the quadratic potential, scaling invariance is no longer available. However, in this paper we are able to obtain a lower bound for the lifespan of the solution to (1.1), based on normal form methods, which has been used to get long time existence results for the corresponding Klein–Gordon equation

$$\partial_t^2 u - \Delta u + |x|^2 u + m^2 u = u^2 \tag{1.5}$$

with $m > 0$ fixed outside a zero measure subset (see [11]). The idea is to perturb the $\mathscr{H}^s(\mathbb{R}^d)$-norm of the solution in a way that the low frequency part would disappear, up to high order terms, when computing its time derivative. This would magnify the smallness of data, allowing one to get a longer existence time interval than the one obtained by local theory. The small divisor which appears in this normal form procedure is a linear combination of 3 eigenvalues because the nonlinearity is quadratic. This is also the reason why there is not a generic parameter like the mass for the Klein–Gordon equation (for instance like in [11]) thanks to the spectrum property of $-\Delta + |x|^2$. If one considers the nonlinearity u^N, one should estimate a small divisor with a linear combination of $N + 1$ eigenvalues and that is why it seems to be very difficult to get a long time result "without a generic parameter" for a nonlinearity with a high degree.

Let $\mathscr{H}^s(\mathbb{R}^d) \times \mathscr{H}^{s-1}(\mathbb{R}^d)$ be endowed with the norm $\|(f, g)\|_{\mathscr{H}^s(\mathbb{R}^d)\times\mathscr{H}^{s-1}(\mathbb{R}^d)} = \|f\|_{\mathscr{H}^s} + \|g\|_{\mathscr{H}^{s-1}}$. By local existence theory, problem (1.1) admits a unique solution defined on the time interval $|t| \le c\epsilon^{-1}$ for any (f, g) in the unit ball of $\mathscr{H}^s(\mathbb{R}^d) \times \mathscr{H}^{s-1}(\mathbb{R}^d)$, provided s is large enough and $\epsilon > 0$ is small enough. The main result of this paper is the following:

Theorem 1.1. *Let d be an odd integer. There exist $\epsilon_0 > 0, c > 0, s_0 > 0$ such that for any natural number $s \ge s_0$, any $\epsilon \in (0, \epsilon_0)$, any real-valued-function pair (f, g) belonging to the unit ball of $\mathscr{H}^s(\mathbb{R}^d) \times \mathscr{H}^{s-1}(\mathbb{R}^d)$, problem (1.1) has a unique solution*

$$u \in C^0((-T_\epsilon, T_\epsilon), \mathscr{H}^s(\mathbb{R}^d)) \cap C^1((-T_\epsilon, T_\epsilon), \mathscr{H}^{s-1}(\mathbb{R}^d))$$

with $T_\epsilon \ge c\epsilon^{-\frac{3}{2}}$ if $d \ge 3$ and $T_\epsilon \ge c\epsilon^{-\frac{19}{12}(1-\rho)}$ for any $\rho > 0$ if $d = 1$. Moreover, (u, u_t) is uniformly bounded by $K\epsilon$ for some constant $K > 0$ independent of (f, g) in $\mathscr{H}^s(\mathbb{R}^d) \times \mathscr{H}^{s-1}(\mathbb{R}^d)$ on $(-T_\epsilon, T_\epsilon)$.

Remark 1.1. As in [11], there are two different results whether $d = 1$ or $d \ne 1$. This is because the sequence of the Hermite function (h_n) (normalized in $L^2(\mathbb{R})$) fulfills $\|h_n\|_{L^\infty} \le Cn^{-\frac{1}{6}}$, which may be seen as a slight dispersive effect, while if $d \ge 2$, such an estimate does not hold for the spectral projector (see Corollary 3.2 in [5]). Therefore, one gets better estimate in Proposition 2.2 in dimension 1 and finally a longer time.

Remark 1.2. By the same analysis, the result of the above theorem holds for the following equation for d even:

$$(\partial_t^2 u - \Delta u + |x|^2 u + u) = u^2, \quad x \in \mathbb{R}^d.$$