Nonlinear Analysis 130 (2016) 214–228

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Characterization of two-weighted inequalities for multilinear fractional maximal operator

Mingming Cao, Qingying Xue*

School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, People's Republic of China

ARTICLE INFO

Article history:
Received 9 March 2015
Accepted 3 October 2015
Communicated by Enzo Mitidieri

Keywords:
Multilinear fractional maximal operator
Two-weight inequalities
Tent space
$S_{(\vec{p},q)}$ condition
$B_{(\vec{p},q)}$ condition

ABSTRACT

In this paper, we restudied the two-weight problem of multilinear fractional maximal operator $\mathcal{M}_\alpha$. First, we gave a characterization of two-weight inequalities for $\mathcal{M}_\alpha$ related to a multilinear analogue of Sawyer's two-weight condition $S_{(\vec{p},q)}$, which essentially improved and extended some known results before. This was done mainly by using the technique of the well known atomic decomposition of tent space. Then we obtained the strong boundedness of $\mathcal{M}_\alpha$ associated with multiple weight $A_{(\vec{p},q)}$ class, which removed the power bump condition assumed in the known results before. Finally, a new two-weight $B_{(\vec{p},q)}$ condition was introduced and the two-weight inequality of $\mathcal{M}_\alpha$ with $B_{(\vec{p},q)}$ condition was established.

1. Introduction

The two-weight problem for linear operators originated in the works of Muckenhoupt and Wheeden [14–16] in the 1970s. The general question is to find a necessary and sufficient condition for a pair of unrelated weights w and v for which the following estimate holds

$$\big\|T(f)\big\|_{L^q(v)} \le A\big\|f\big\|_{L^p(w)},$$

for a finite constant A independently of f. In 1982, Saywer [21] showed that the fractional maximal operator

$$M_\alpha f(x) := \sup_{Q\ni x} \frac{1}{|Q|^{1-\alpha/n}} \int_Q |f(y)|dy$$

is bounded from $L^p(w)$ to $L^q(v)$ if and only if (w,v) satisfies that

$$[w,v]_{S_{(p,q)}} := \sup_{Q\in\mathcal{Q}} \frac{\left(\int_Q M_\alpha(\sigma\chi_Q)^q v dx\right)^{1/q}}{\sigma(Q)^{1/p}} < \infty,$$

* Corresponding author.
E-mail addresses: m.cao@mail.bnu.edu.cn (M. Cao), qyxue@bnu.edu.cn (Q. Xue).

http://dx.doi.org/10.1016/j.na.2015.10.004

where $\sigma = w^{1-p'}$, $0 \leq \alpha < n$, $1 < p < n/\alpha$, $\mathcal{Q}$ is the family of all cubes in $\mathbb{R}^n$ with sides parallel to the axes. In 1984, Saywer [22] gave a sufficient and necessary condition for two-weight weak type inequality of the fractional integral operator I_α, which was defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

They proved that the following inequality

$$v\big(\{I_\alpha f(x) > \lambda\}\big) \leq \frac{A}{\lambda^q} \|f\|^q_{L^p(w)},$$

holds if and only if

$$\int_Q \Big(I_\alpha(\chi_Q v)(x)\Big)^{p'} w^{1-p'} dx \leq B\Big(\int_Q v dx\Big)^{q'/p'} < \infty, \quad \text{for all cubes } Q \in \mathbb{R}^n.$$

In 1988, as for the fractional integral I_α and a kind of more general convolution operator with radial kernel decreasing in $|x|$, Saywer [23] obtained a characterization for two-weight strong type inequality.

Inspired by the above results, the theory of two-weighted inequalities developed rapidly. Among such achievements are the nice works of Lacey and Li [7], Perez and Rela [19], Sawyer and Wheeden [24], and Wheeden [25]. Much later, the interests were focused on determining the sharp dependence of the $L^p(w) \to L^q(v)$ operator norm in terms of the relevant constant involving the weights, see [6,8,12] for more details. Moreover, in 2009, Moen [12] extended Sawyer's result by proving that

$$\|M_\alpha\|_{L^p(w)\to L^q(v)} \asymp [w, v]_{S_{(p,q)}}. \tag{1.1}$$

However, Saywer's condition is often difficult to verify in practice, since it involves the maximal operator. Thus it is necessary to look for other simple sufficient conditions. The first attempt was made by Neugebauer [17] in 1983. He gave a sufficient condition closely in spirit to the classical A_p condition: if (w, v) satisfies

$$\sup_{Q\in\mathcal{Q}} \Big(\frac{1}{|Q|}\int_Q v^{pr} dx\Big)^{\frac{1}{pr}} \Big(\frac{1}{|Q|}\int_Q w^{-p'r} dx\Big)^{\frac{1}{p'r}} < \infty, \quad \text{for some } r > 1, \tag{1.2}$$

then $\|Mf\|_{L^p(v^p)} \lesssim \|f\|_{L^p(w^p)}$. Later, in 1995, Pérez [18] improved condition (1.2) by

$$\sup_{Q\in\mathcal{Q}} \Big(\frac{1}{|Q|}\int_Q v^{p} dx\Big)^{\frac{1}{p}} \Big(\frac{1}{|Q|}\int_Q w^{-p'r} dx\Big)^{\frac{1}{p'r}} < \infty.$$

The two-weight problem for multilinear operators has also been studied recently. Let us first recall some related known results. In 2009, the multilinear fractional type maximal operator

$$\mathcal{M}_\alpha(\vec{f})(x) = \sup_{\substack{Q\ni x\\ Q\in\mathcal{Q}}} \prod_{i=1}^m \frac{1}{|Q|^{1-\frac{\alpha}{mn}}} \int_Q |f_i(y_i)|\, dy_i, \quad \text{for } 0 \leq \alpha < mn,$$

was first introduced and studied by Chen and Xue in [4], and also simultaneously defined and studied by Moen in [13]. Specially, when $\alpha = 0$, $\mathcal{M}_0$ is coincide with the multilinear maximal function and will be denoted by $\mathcal{M}$, which was introduced by Lerner, Ombrosi, Pérez, Torres and Trujillo-González in [9]. To study two-weight inequality, Moen [13] introduced the following two-weight condition

$$[\vec{w}, v]_{A_{(\vec{p},q)}} := \sup_{Q\in\mathcal{Q}} |Q|^{\frac{\alpha}{n}+\frac{1}{q}-\frac{1}{p}} \Big(\frac{1}{|Q|}\int_Q v dx\Big)^{\frac{1}{q}} \prod_{i=1}^m \Big(\frac{1}{|Q|}\int_Q w_i^{1-p_i'} dx\Big)^{\frac{1}{p_i'}} < \infty. \tag{1.3}$$